Curvature criteria in surface reconstruction

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1 Introduction

In this paper we discuss the problem of reconstructing surfaces from discrete scattered (irregularly distributed) data. One of the usual ways to pre-process the data is to construct first an initial triangulated polyhedral surface that spans all given data points. The data points will then be the vertices of the obtained triangulated surface.

We refer to the surface, which we want to reconstruct, as to the *source surface*. A polyhedral (triangulated) surface, reconstructed from the given discrete data set, can be considered as a *polyhedral model* of the source surface. Depending on the choice of the connections among the vertices, we can construct many polyhedral models for a given data set V.

The key question is:

- How to determine the measure of adequacy of a polyhedral model with respect to the source surface?

Informally speaking, the obtained polyhedral model is adequate if we can recognise the source surface from the model, retrieve its shape, visualise it etc. A surface possesses *topological* and *metric* properties. The metric properties are those that depend on the extent and details of the shape of the features in the structure under study. The notion of the shape (not well–defined) is directly related to the concept of curvature (well–defined). There exist several types of curvature for a surface, the main ones are the Gaussian and Mean curvatures. Topology concerns with those properties of the shape that do not change under deformation (one-to-one and bicontinuous).

We can say that our model is adequate if, on the base of this model, we can determine the topological type of the source surface, and/or can extract its basic geometric characteristics by computing, for example, its curvatures.

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In this communication we assume that possible polyhedral models preserve the main topological characteristics of the source surface, *i.e.*, its *genus* and *orientability*. Therefore, if we could estimate the curvatures (up to an admissible tolerance) of the source surface from a polyhedral model, then we could consider this model as adequate.

If we know the analytical representation of the surface, then we can easily compute its curvatures by using the well–known formulae of differential geometry [7]. These formulae require high order differentiability of the surface (at least the second order) and, therefore, are not applicable to polyhedral surfaces, which are of C^0–class. Nevertheless, analogues of curvatures for polyhedral surfaces exist and easily can be computed [2, 6]. The problem is how well these analogues approximate the curvature values of the source surface, which can be smooth or non–regular. The number of polyhedral models is huge, many of them might be quite inadequate.

We investigate which polyhedral model best approximates the curvatures of the source surface. We can change one polyhedral model to another just by swapping an edge [3]. This operation can be considered as an optimisation procedure, and, hence, the problem to find an adequate polyhedral model can be transformed into the problem of optimisation of an initial polyhedral model.

In Section 2 we present some notions and definitions, concerning triangulations, polyhedral surfaces and curvatures. We give also a brief overview of the theory of *manifolds with a polyhedral metric*, or simply *polyhedral metrics*, which form the basic part of the theory of non-regular surfaces developed by A.D. Aleksandrov and his school in Russia [14]. Polyhedral surfaces are typical examples of polyhedral metrics.

In Section 3 we discuss how the notions, presented in Section 2, can be used in Surface Reconstruction and other applications where one deals with 3D modelling from discrete data. We present several geometric variational criteria to obtain an adequate polyhedral model. All these criteria are related to minimisation of one or another curvature, defined for a polyhedral surface. Most of these criteria have been introduced by the author and her long-term collaborator Ruud van Damme [2, 3].

The last topic which we discuss is the relation between the data and the adequacy of the model. Our model can be optimal among all polyhedral models with respect to the given criterion, but still non–adequate, because the given data do not sufficiently represent the source object. Therefore, the second key question is:

- How to determine whether the data are sufficient?

We discuss an approach to answer this question, by using some notions of the critical point theory and the theory of non–regular surfaces [4, 14].

2 Basic notions and definitions

In this section some main geometric notions and definitions are given. We presume that the reader is familiar with basic notions of differential geometry. We give all necessary explanations whenever we consider this appropriate, otherwise we refer the reader to the relevant literature.

2.1 Triangulation

Triangle meshes (shortly, triangulations) provide an essential step in the spatial analysis of the data and are commonly used for representing surfaces in 3D. Given a set of vertices sampled from a source (in general, smooth) surface, a triangular mesh (triangulation) with these vertices serves as a representation (approximation) of the sampled surface. The quality of the approximation obviously depends on the particular choice of the triangulation. On the base of the triangulation a PLIS (Piecewise Linear Interpolating Surface)) is uniquely determined.

The following definition of triangulation is standard:

Definition 2.1 *A triangulation T is a collection of triangles, that satisfies the following properties:*

1. *Two triangles are either disjoint, or have one vertex in common, or have two vertices and consequently the entire edge joining them in common.*

2. *T is connected.*

We are interested only in compact surfaces. In this case a triangulation T consists of a finite number of triangles, and two following conditions are valid [12]:

1. Each edge is an edge of exactly two triangles.

2. For every vertex V of a triangulation T, we may arrange the set of all triangles with V as a vertex in cyclic order, $T_0, T_1, ..., T_{n-1}, T_n = T_0$, such that T_i and T_{i+1} have an edge in common for $0 \leq i \leq n-1$.

The last condition means, that our triangulated surface is a manifold, *i.e.*, the neighbourhood of every point, as well as a vertex, is topologically the same as the open unit ball in $\mathbf{R^2}$. The first condition of Definition 2.1 excludes any intersection. Actually, the above–mentioned definition is the definition of an *abstract* *triangulation* of an abstract surface [10]. Real surfaces can have self–intersections and/or have singularities, in which a surface is not a manifold. Therefore, we should distinguish between the abstract topological representation of a (triangulated) surface S (as a 2-dimensional *manifold*) and its realisation in 3-space (as an image of this "canonical" manifold).

Given the data, we can obtain many triangulations depending on the connections between vertices. We assume that each triangulation preserves main topological characteristics of the underlying source surface: its genus and orientability. Then each triangulation determines a PLIS and we can switch from one PLIS to another just by swapping an edge [2, 3]. Those surfaces are called PL (piecewise–linear) homeomorphic surfaces. They all have the given data set as vertices, they are continuous (C^0–class), and can be also defined as manifolds with polyhedral metrics. For manifolds with polyhedral metrics analogues of the essential geometric features of smooth manifolds are determined.

2.2 Polyhedral metrics

A manifold with a polyhedral metric (MPM) is a metric space every point of which has a neighbourhood isometric to the lateral surface of a cone. This neighbourhood may be a plane domain. Indeed, all the points of a space with polyhedral metric, except for a discrete set of points called vertices, have a plane neighbourhood. We can define similarly manifolds with boundary. The points on the boundary have neighbourhoods isometric to a plane angular sectors (of arbitrary angle).

The most typical examples of MPM are the surface of any polyhedron in $\mathbf{R^3}$ as well as its development into plane polygons. From the definition it follows that a ball $U(a,r)$ in *MPM*, when r is sufficiently small, is isometric either to the plane circle of radius r with the centre in a, or to a surface of the cone with the vertex in a and the generator of length r. The value

$$\theta(a) = \frac{S(r)}{r},$$

where $S(r)$ stands for the length of the boundary of $U(a,r)$, r is sufficiently small, is called the full angle $\theta(a)$ at the point $a \in N$. (Obviously, when r is sufficiently small, the value $S(r)/r$ is constant). Points a, in which $\theta(a) \neq 2\pi$, are called vertices of a polyhedral metric. It is clear, that in any compact domain of a manifold with a polyhedral metric, the number of vertices is finite, and the set of all vertices is not more than countable. We can develop such a manifold on a plane. The development will consist of plane polygons with rules for identifying their sides. If we triangulate a two–dimensional Riemannian manifold M (informally, a manifold where the distance between two points is defined) by means of shortest arcs and replace each simplex of this triangulation by a plane triangle with sides of the same lengths, we obtain a manifold with a polyhedral metric.

The sequence of such manifolds will converge to M as the size of the triangles tends to zero [1, 14].

Another important feature of polyhedral metrics is that they form the basic part of the theory of *non–regular manifolds with bounded curvature*, developed by A. D. Aleksandrov and his school in Russia. Most of the surfaces encountered in applications dealing with 3D modelling, are irregular and even not smooth (human organs, biological structures, Earth's regions in terrain modelling). Classical differential geometry deals only with regular, or at least with smooth surfaces. Roughly speaking, it deals with pieces of cylinders, or cones, but not with whole cylinders or cones, because of their having singularities. A surface "in its whole" is also the subject of the study in differential geometry in the large.

In the second part of the last century the study of surfaces in the large as well as that of irregular surfaces has attracted much attention and development in pure mathematics. Whereas differential geometry in the large was developed both in the West and in Russia, the theory of non–regular surfaces has been mainly developed in Russia. Especially the manifolds with bounded curvature (*MBC*) were extensively studied. The curvature of most non-regular surfaces encountered in 3D modelling is bounded (if they don't have "needles" of zero diameters). Any MBC of bounded curvature is, in a certain sense, the limit of manifolds with polyhedral metrics [14].

This property might be very useful in obtaining a "superior" model of the source surface. One might require that the model would be of reasonable smoothness, if the source surface is smooth, or would extract all important singularities of a non–regular source surface and be sufficiently smooth to represent the smooth domains of the source surface. In order to do this one can create a hierarchy of polyhedral models starting from an initial model and proceeding to the final model, which will represent the source surface within a prescribed or desired tolerance. We are conducting preliminary investigations in this research direction now.

2.3 Curvatures revised

A triangular mesh (synonyms: a 2.5D triangulation, a triangulated polyhedral surface), which vertices are given discrete data points sampled from the source surface (smooth or not) are C^2–differentiable everywhere besides the vertices themselves and the edges (connections between the vertices), where it is C^0 only. The essential geometric features of any smooth surface are its curvatures. A notion of curvature is strictly related to the notion of *angle*, and this gives us a possibility to determine analogues of curvatures also for polyhedral surfaces viewed as polyhedral metrics. We can easily see, as triangles are flat domains, that curvatures are concentrated exactly around the vertices and along the edges. Below we give an overview of the curvatures, which we can determine for a triangulated polyhedral surface. For precise definitions of the corresponding notions of classical differential geometry see [7].

Let P be a polyhedral (triangulated) surface. The *total angle* $\theta(x)$ around the vertex $x \in P$ is the sum of angles of all the plane polygons incident to x. If $x \in P$ is not a vertex, we define $\theta(x) = 2\pi$. For any point $x \in P$ the curvature ω is defined as $\omega(x) = 2\pi - \theta(x)$. Only for vertices we have $\omega(x) \neq 0$ (see Fig. 1).

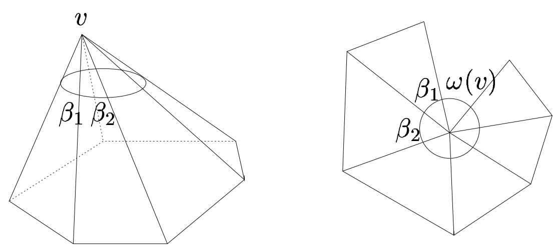


Figure 1: Curvature around a vertex: $\omega(v) = 2\pi - \sum \beta_i = 2\pi - \theta(v)$.

ω can be regarded as a 0–dimensional distribution. The expression ω is also known as the *angle deficit*. For a point y on the boundary the curvature is defined as follows

$$\omega(y) = \pi - \theta(y) \tag{1}$$

By the curvature $\omega(U)$ of any set $U \in P$ we call the sum of the curvatures of the vertices of P, which lie in U. The sum of curvatures of those vertices of U, in which the curvature is positive, is called the positive part $\omega^+(U)$ of the curvature of U, the sum of the absolute values of the curvatures of those vertices in U, where the curvature is negative, is called the negative part $\omega^-(U)$ of the curvature $\omega(U)$ respectively. Hence,

$$\omega(U) = \omega^+(U) - \omega^-(U).$$

The value

$$\Omega(U) = \omega^+(U) + \omega^-(U)$$

is called the *variation of the curvature* or the *absolute curvature* of the set $U \subset P$.

In the case of a 2–dimensional smooth (of order ≥ 2) manifold M the above expressions coincide with the following ones:

$$\omega^+(Q) = \int_E K^+ dA, \omega^-(Q) = \int_Q K^- dA,$$

$$K^+ = \max\{0, K\}, K^- = \max\{0, -K\},$$

where K is the Gaussian curvature, $Q \subset M$, dA – the surface element of M. And $\omega(Q)$ is known as *integral (Gaussian) curvature* of the set $Q \subset M$.

However, for polyhedral surfaces $\Omega(U)$ is not really an analogue of corresponding expression for smooth surfaces. In the smooth case points with positive and negative are isolated (around a point with positive (negative) curvature there exists always a neighbourhood, where the sign of curvature is preserved). In the "polyhedral case" positive and negative curvatures might be "glued" together. See, for example, Fig. 2:

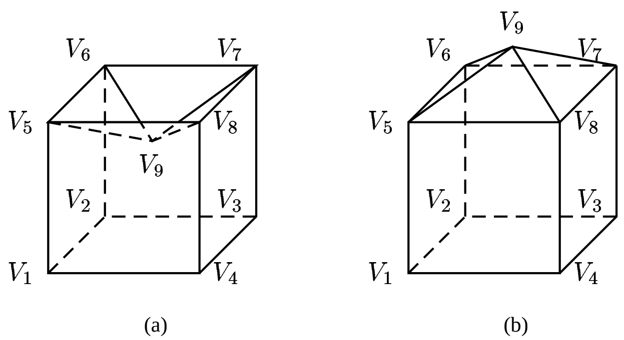


Figure 2: Two "distorted" cubes *Cub1* and *Cub2* with $\Omega(Cub1) = \Omega(Cub2)$.

In this figure you can see two surfaces, both originated from a cube. In the first one the point is added "inside" the cube, just producing a concave region, and in the second surface the point is added "outside" the cube, producing a convex region. Both points are the mirror images of each other with respect to the plane $V_5V_6V_7V_8$. In both cases the curvatures around the corresponding vertices are positive and equal, and $\Omega(Cub1)(= \Omega(Cub2) = 4\pi$. Note it is not necessary to triangulate the surfaces, because the faces are flat domains. *Cub2* is a convex figure, but *Cub1* is not. "Saddle" and "convex" ("concave") domains around vertices V_5, V_6, V_7,and V_8 are "glued" together and we need to isolate them. Below you can see how to do this.

We deal with a triangulated surface (a triangulation), we will refer to a triangulation as to $\triangle$ instead of P, which denotes a general polyhedral surface.

Let us denote by Str(v) (*star*) of vertex v the union of all the faces and edges that contain the vertex, and by *link* of the star (the boundary of the star) – the union of all those edges of the faces of the star Str(v) that are not incident to v.

Then for any point of $\triangle$:
$x \in \triangle$: $\omega(x) = 2\pi - \theta(x)$. Only for vertices $\omega(x) \neq 0$.
2. The *positive (extrinsic) curvature* $\omega^+(v_i)$.
Suppose, some (local) supporting plane of $\triangle$ passes through vertex v_i. Then this vertex lies on the boundary of the convex hull of Str(v). We denote the star of v_i in the boundary of this convex hull by Str$^+(v_i)$ and will call it the *star of the convex cone* of a vertex (or, simply, the convex cone of a vertex, if it does not lead to ambiguities). The curvature $\omega^+(v_i)$ of Str$^+(v_i)$ is called the *positive (extrinsic) curvature* of v_i. If there is no supporting plane through v_i then $\omega^+(v_i)$ is equal to zero by definition.
3. The *negative (extrinsic) curvature* $\omega^-(v_i)$.
$\omega^-(v_i) = \omega^+(v_i) - \omega(v_i)$.
4. The *absolute (extrinsic) curvature* $\hat{\omega}(v_i)$.
$\hat{\omega}(v_i) = \omega^+(v_i) + \omega^-(v_i)$

We can isolate the following types of vertices:

• *Convex vertices*: $\omega(v_i) = \omega^+(v_i) = \hat{\omega}(v_i)$, $(\omega^-(v_i) = 0)$.
Geometrically it means that Str(v) coincides with Str$^+(v_i)$.

• *Saddle vertices*: $\hat{\omega}(v_i) = \omega^-(v_i) = -\omega(v_i)$ $(\omega^+(v_i) = 0)$.
The Gaussian curvature ω of a saddle vertex is less than zero and there exists no supporting plane.

• *Mixed vertices*:
1) $\omega(v_i) > 0,\ \omega^+(v_i) > \omega(v_i)$
or
2) $\omega(v_i) < 0,\ \omega^+(v_i) > 0$.

In Fig. 3 examples of all three types of vertices are presented.

You can see a mixed vertex and its correspondent convex star in Fig. 4.

The *total absolute (extrinsic) curvature* $\hat{\Omega}_{abs}(\triangle)$ of a triangulation $\triangle$ is given by the following expression:

$$\hat{\Omega}_{abs}(\triangle) = \sum_{v_\alpha \text{ convex}} \omega^+(v_\alpha) + \sum_{v_\beta \text{ saddle}} \omega^-(v_\beta) + \sum_{v_\gamma \text{ mixed}} (\omega^+(v_\gamma) + \omega^-(v_\gamma)). \quad (2)$$

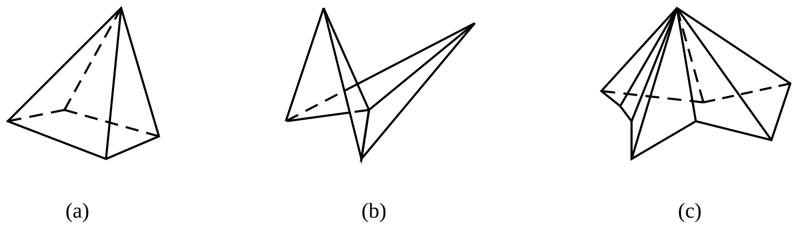


Figure 3: Types of vertices: (i) Convex (ii): Saddle (iii) Mixed.

Figure 4: A mixed vertex (i) and its convex star (ii).

For any triangulation convex, saddle and mixed vertices form three disjoint subsets of the vertices of the triangulation, and their union is the data set itself.

The notion of *mean curvature* can be also defined for polyhedral metrics. It is also based on the concept of *angle*. Roughly speaking, mean curvature defines how much the surface is "bent" in space. For smooth surfaces mean curvature at some point w is given by means of the formula:

$$H = \frac{k_1 + k_2}{2}, \tag{3}$$

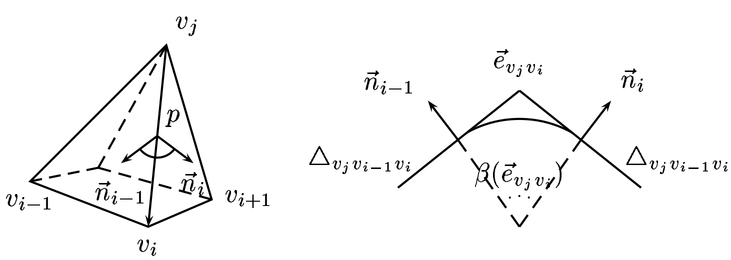
where k_1 and k_2 are *principal curvatures* at w [7]. However, principal curvatures at w can be replaced by curvatures of curves determined by any two mutually orthogonal directions at this point [13].

We use this fact in order to determine an analogue of mean curvature for a polyhedral surface P. It is easy to see that polyhedral surface P is "bent" along an edge. For a point on the edge we choose two directions – the direction along this edge ($\bar{e}$) and the direction perpendicular to it ($\bar{\perp}_{\bar{e}}$). The line, determined by $\bar{e}$ is a straight line, so its curvature is equal to zero, and the line, defined by $\bar{\perp}_{\bar{e}}$, considered as a plane line, has a rotation (curvature), equal to the exterior angle between the faces adjacent to the edge $\bar{e}$. If we want to take into account the sign we can make the following definition of the mean curvature H:

> $H(e)$ is equal to the half of the oriented exterior angle between the faces adjacent to $\bar{e}$ and zero otherwise.

The sign of H depends on the orientation of the surface P. H can be regarded as a 1–dimensional distribution.

We can give another justification of the above–written definition. Suppose we isolate edge $e_{v_j v_i}$ that connects vertices v_j and v_i; $e_{v_j v_i}$ belongs to the star of vertex v_j. Let us inscribe in the dihedral angle defined by the given edge $e_{v_j v_i}$ a cylinder. Its radius is equal to R. The value of R can be arbitrary. Now, let us cut the dihedral angle in some point $p \in e_{v_j v_i}$ by the plane perpendicular to the edge. We obtain a plane angle equal to the given dihedral angle and a circumference o of the radius R inscribed in it. The curvature of this circumference is equal to $1/2R$. It is equal to one of the principal curvature of the initial inscribed cylinder (in the direction perpendicular to the generator of the cylinder which is parallel to the direction along the edge e). This curvature depends on the radius R. However, the integral curvature of the piece of the circumference containing between the points of contact of this circumference with the sides of the plane angle does not depend on the radius and is equal to the exterior dihedral angle $\beta(\vec{e}_{v_j v_i})$ defined by the edge $e_{v_j v_i}$ (see Fig. 5).



(a) (b)

Figure 5: Analogue of mean curvature, determined along edge $\vec{e}_{v_j v_i}$.

Integral mean curvature (IMC) determined for domain $U \subset P$ is defined as follows:

$$H(U)_{U\cap P} = \sum_e H(e)\cdot \text{length}(e\cap U)$$

where the sum ranges over all edges e of U. The total absolute Mean curvature is defined then as

$$H(U)_P = \sum_e H(e)\cdot \text{length}(e),$$

where the sum ranges over all edges e of P.

2.4 Critical points and curvatures

One of the most remarkable results in "geometry in the large" establishes a connection between the notions of (total) curvature of a surface and that of critical points for non–degenerate height functions, determined on this surfaces.

A height function on a smooth closed surface S can be defined as follows.

Definition 2.2 *Let Σ denote the unit sphere in $\mathbf{R^3}$. For any unit vector $\xi \in \Sigma$ the linear height function $\ell_\xi : S \to \mathbf{R}$ is defined by $\ell_\xi(x) = \langle p, \xi\rangle$, where $\langle\ ,\ \rangle$ denotes the Euclidean inner product.*

This function has some critical point. Geometrically it means that the tangent plane to the surface S at a critical point p is perpendicular to ξ. All other points are called *ordinary points* for ℓ_ξ.

To a critical point there corresponds a number, or the *index*, denoted by $i(p,\xi)$, which, in a certain sense, gives the idea about the "complexity" of a critical point. In the case of a local maximum or minimum $i(p,\xi) = 1$, and in the case if p is a (non–degenerate) saddle point, $i(p,\xi) = -1$. The classical result for a smooth surface S embedded in $\mathbf{R^3}$ is that for almost every unit vector ξ on 2-dimensional sphere Σ (except for a set of measure zero on Σ), the height function ℓ_ξ has only finitely many critical points, and, moreover, for almost all ξ, the height function has as critical points only local maxima and minima, and non-degenerate saddle points. Under these conditions, the Critical Point Theorem states [4]:

$$\sum_{p critical for \xi} i(p,\xi) = \chi(S) \tag{4}$$

$\chi(S)$ is Euler–Poincaré characteristics.
For example, a height function for a "vertical" torus of revolution has four critical points, a maximum, a minimum, and two (non-degenerate) saddle points (see Fig. 6).

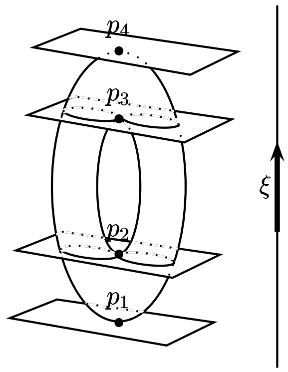


Figure 6: The height function for a vertical torus

The total curvature K of a surface can be expressed in terms of indices $i(p, \xi)$ (for proof, see, for example, [4]):

$$K(U) = \frac{1}{2} \int_{\Sigma} \sum_{p \in U} i(p, \xi) d\Omega, \tag{5}$$

where the integrand is well defined as almost every ℓ_ξ has at most finitely many critical points.
Hence, the expression for absolute total curvature can be written as:

$$\int_S |K| dA = \int_{\Sigma} \frac{1}{2} \sum_{p \in S} |i(p, \xi)| d\Omega, \tag{6}$$

where the right-hand integrand is well-defined almost everywhere (except on measure zero).
The index of a critical point admits the geometric interpretation of critical points which is also valid for polyhedral surfaces. Around any point $a \in M$, where M is a 2–dimensional manifold we can choose a "small disc neighbourhood" $O(a)$. If point q is ordinary for the height function ℓ_ξ, then the tangent plane at q is not perpendicular to ξ and hence meets a "small circle" $\bar{O}(q)$ in exactly two points. If at p there exists a local maximum (minimum) the tangent plane does not meet a "small circle" at all, at a non-degenerate saddle point it intersects $\bar{O}(p)$ in four distinct points. Then the definition of the index of a point can be the following:

$$i(p, \xi) = 1 - \frac{1}{2} N(p, \xi), \tag{7}$$

where $N(p,\xi)$ is the number of points in which the plane through p perpendicular to ξ will intersect a "small circle" about p on M. In this form the definition is valid for smooth surfaces, if one defines $i(p,\xi) = 0$ for an ordinary point, as well as for polyhedral surfaces. For polyhedral surfaces, however, one has to consider *general height functions*. A height function ℓ_ξ on a unit sphere Σ is said to be general for the polyhedral surface Q if $\ell_\xi(v) \neq \ell_\xi(u)$ whenever v and u are distinct vertices of Q. It is easy to see that almost all ℓ_ξ are general. For polyhedral surfaces one can use the polygon which is the boundary in Str(v_i) *i.e.*, Lnk(v_i), instead of a small circle. Then $N(p,\xi)$ is equal to the number of times the plane P_{v_i} through v_i and perpendicular to ξ, intersects Lnk(v_i). This number is equal to the number of triangles in Str(v_i), which P_{v_i} divides in two non–void parts. It is not difficult to show that this number is always even.

The formulae analogous to the formulae 5 and 6 hold for $\omega(\triangle)$ and for $\Omega_{abs}(\triangle)$.

Using the above–mentioned concepts, we can see that for *Cub2* in all directions ξ of general height function ℓ_ξ will have exactly two critical points: a local maximum and a local minimum, which is not true for *Cub2*.

3 Test cases and problems

Any surface, also a polyhedral one, possesses intrinsic and extrinsic geometry. Intrinsic properties are those that are determined by measurements along a surface S itself, while extrinsic ones depend only on the way S lies in the ambient space. A triangulation, as a polyhedral surface, possesses also intrinsic and extrinsic geometric properties. The geometry of a triangulation might be very different from the geometry of the underlying surface. If we could determine the triangulation, which would approximate the surface in the "best" possible way, then we could already extract the geometrical properties of the surface directly from the triangulation. This primal "sketch" of the surface, in turn, could give very useful information for further elaboration.

3.1 Shape "extraction"

The idea is that two main curvatures associated with a surface, describe sufficiently good the shape of a surface. The Gaussian curvature tells whether a region is elliptic, hyperbolic or parabolic. The mean curvature tells whether a region is "full" or "hollow". The signs of Mean and Gaussian curvature yield eight 'basic' surface region's types (in terms of terrain modelling [5]):

$H > 0, K > 0$	'peak' surface region (convex)
$H = 0, K = 0$	flat surface region
$H < 0, K > 0$	'pit' surface region (concave)
$H = 0, K < 0$	minimal surface region
$H > 0, K = 0$	'ridge' surface region
$H > 0, K < 0$	'saddle ridge' surface region
$H < 0, K = 0$	'valley' surface region
$H < 0, K < 0$	'saddle valley' region.

The methods of the theory of non–regular surfaces, in particular, of *polyhedral metrics*, as they have "discrete" analogues of curvatures and of other geometrical quantities, might give a new insight into geometry processing as well as into the surface reconstruction problem as a whole. However, as we said above, there are many triangulations for the same data set taken from the source surface. Let us consider two triangulations of the same data set that are presented in Fig. 7.

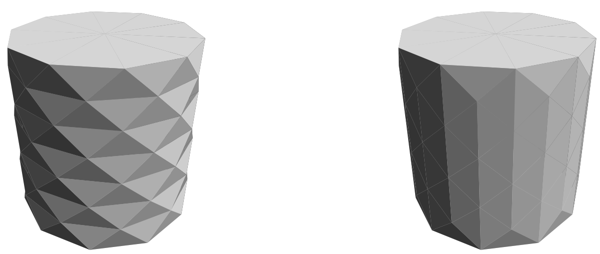


Figure 7: Left: Delaunay traingulation; Right: Tightest triangulation

The data were taken uniformly from the surface of a cylinder, but we can recognise the cylinder only in the body on the right side of the picture. Two triangulations are different, and their curvatures are also different. If we want to get some preliminary sketch of geometric properties of the source surface (the cylinder) : for example, to determine approximately its shape, - it has no sense to use for this purpose the left triangulation, but the analysis of the second one can be very useful. It gives a good representation of the source surface.

The question is: how to obtain the second triangulation?

3.2 Optimality criteria

In general, different requirements to an optimal triangulation are considered, and on the base of these requirements one chooses an optimal criterion. For some overview see [3].

We say here only some words about the curvature criteria, introduced by L. Alboul and van Damme.

Let us again consider Fig. 7. The right triangulation is a convex triangulation, the cylinder is a convex figure. From global differential geometry we know that the total absolute Gaussian curvature for surfaces of genus 0 reaches the minimal value 4π on convex surfaces. Thus the idea is to use the minimisation of Ω_{abs} as a criterion of optimisation. However, minimisation of Ω_{abs} has a deeper meaning. It reaches its minimal value on tight surfaces, which can be of different genera [11]. Triangulations with minimal Ω_{abs} are called *Tightest triangulations* [3]. See, for example, Fig. 8.

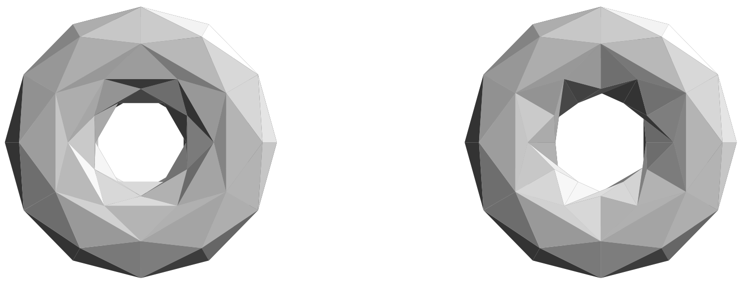


Figure 8: Left: Initial triangultion of a torus; Right: Tightest triangulation

In this picture two triangulations of a torus of revolution are presented. The right triangulation is the Tightest triangulation. You can easily notice that the convex domain of the torus is represented in a similar way in both triangulations, whereas the saddle domain is represented definitely better in the Tightest triangulation.

Other optimality criteria concern the minimisation of various analogues of the Mean curvature [2].

On the base of these criteria other criteria can be determined [9].

Let us note that we know still very little about the triangulations on which geometrical criteria reach their minima. One difficulty is due to the fact that triangulations can be considered as *constrained polyhedral immersions*, because we have a number of fixed points, and the above–mentioned notions have been mostly studied for immersions in the classical sense. Another difficulty is of algorithmic character: we cannot usually guarantee global convergence of the algorithms.

3.3 Sufficiency of data and adequacy of a polyhedral model

In general, one assumes that the given discrete set of points represents the surface in the sense that its most prominent features (creases, curvatures, etc) can be extracted from the data and that the representation as a triangle mesh should not add features not present in the data. From the previous section it follows that all critical points are in the given data. In this case, it seems very appropriate to use optimality criteria based on minimising one or another (discrete) curvature.

Let us, however, consider Fig.9.

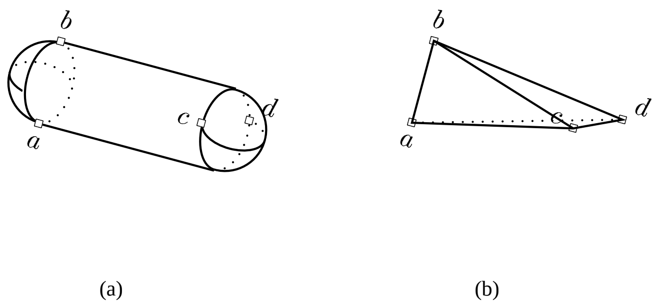


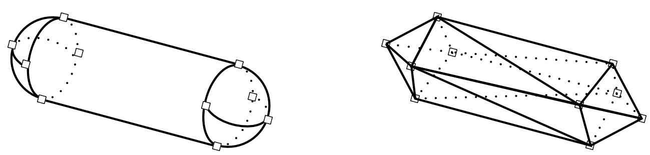
Figure 9: Data and a polyhedral model: (a) Source surface with four data points; (b) A polyhedral model.

At the picture we see a cylinder with two semispheres ($C2S$) and four data points. We can reconstruct a polyhedral model in two ways. One way is just to connect the points, keeping in mind that this is a triangulation, and the other is to use Aleksandrov's approach. Aleksandrov's approach consist of drawing first geodesics (shortest arcs) on the surface, then of developing geodesic triangles in the plane, and, finally, of reconstructing from this development a polyhedron. In the latter case we need to know the source surface *a priori*. But this method can help us to understand better which and how many data points we might need. First of all, the domain abc on $C2S$ (where geodesic ab passes through the top of the semisphere) admits another geodesic ab which connect a and b along the border between the surface of the first semisphere and the surface of the cylinder, and which lies inside this domain. This can bring us to the conclusion that another data point should be on the top of the semisphere. But this case yields a non–proper triangulation (points a and b are connected twice) and we can conclude that we should add more points at the border between the semishere and the cylinder. The same reasoning is valid for the second semisphere and cylinder. This bring us to the idea that we need to have at least eight data points: two on the tops of the semispheres and three points on each border (distributed in a proper way). All these points are critical points for some directions of the height function.

If we don't know anything about the source surface, then we don't know if our data represent the surface sufficiently. Can we still guess that some essential points are missing? In the above–mentioned example it seems we cannot do this. Only the extreme simplicity of the polyhedral model can suggest that something wrong and our data are very sparse.

But if we have more data points?

Let us look now at Fig 10.



(a) (b)

Figure 10: Data and a polyhedral model: (a) Source surface with ten data points; (b) Adequate polyhedral model.

We used the previous reasoning and now we have data points on the tops of the semispheres and reasonable amount of points on two borders. However, our data are still sparse. Actually it is impossible to take all critical points in all directions of the height functions as data points, as it leads to an infinite number of data points. If it would be possible, then by taking some initial triangulation and applying some optimality criterion, we would get the source surface. In the optimisation process all extra critical points (that are specific only for a specific triangulation as a polyhedral surface) would be eliminated.

Nevertheless, the polyhedral model on the right side gives not a bad idea about the source surface. Our conjecture is that at least such critical points must be present that allow us to single out convex, saddle and concave domains of the source object. Then we could add more critical points in each domain in order to obtain a smoother surface or to distribute curvatures more uniformly. For example, we can estimate the curvatures around the vertex on the top of one of the semisphere and around the vertices on the corresponding border. We can notice that the curvature around the vertex on the top is higher then the curvatures around the points on the border. We can aim for decreasing the difference between this curvatures. For this reason we use the notion of *specific curvature* around a vertex. Let us denote by A_{v_i} the sum of the areas of all triangles in Str(v_i). Then the specific curvature is defined as follows:

$$\omega_{spec}(v_i) = \frac{\omega(v_i)}{\frac{1}{3}A_{v_i}}$$

The coefficient $1/3$ is logical because each triangle contains three vertices. We can define *positive and negative specific curvatures* around vertex v_i in a similar way.

Then we can impose some upper and lower bounds on specific curvatures, and on the base of those values add some number of additional points by using, for example, a subdivision scheme [8]. The subdivision scheme might be in the sense of Aleksandrov (see Section 2).

The research in this direction is now in its very beginning.

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