Spatial quasi-isometric mappings as minimizers of polyconvex functionals

*V.A. Garanzha and †N.L. Zamarashkin.
*Computing Center Russian Academy of Sciences, garan@ccas.ru
†Institute of Numerical Mathematics
Russian Academy of Sciences, kolya@bach.inm.ras.ru

Precise control of spatial mappings is a key capability in grid generation, adaptive simulations, geometric modeling, surface parameterization and reconstruction, and many other real-life applications.

Numerically computed grids and mappings should satisfy a set of natural constraints, including local invertibility, which generally precludes the use of simple linear equations or convex functionals as constitutive models. So widely used discrete variational methods for grid generation are based on higly nonlinear nonconvex functionals. The rigorous validations of such variational problems are very scarse. In every mapped grid generation technique one have to answer the following questions:

1) Does the computed grid converge with grid refinement?

2) Is the quality of elements kept with grid refinement?

3) Is the grid stable with respect to input data?

4) Is it possible to have precise control over any property of a grid ?

The first step in answering these questions is to analyze continuous variational problem which is used to build the discrete one. There are several well established approaches to this problem. The first one is based on the theory of harmonic maps. Unfortunately in many practical cases harmonic maps are notoriously singular. Very promising approach is based on the theory of conformal maps. In particular in [1] is was shown that in certain cases conformal mapping is unique and quasi-isometric, meaning that the poitwise quality of mapping is guaranteed. However this approach cannot be applied in spatial case. More general approach was developed using the ideas from hyperelasticity with large deformations. Various functionals were suggested for description of real elastic materials which allow to construct locally invertible mappings in multiple dimensions and in general domains. General existence theory for such functionals was developed in [2] and was shown to reside on the concept of polyconvexity. However major problems with these functionals are that they may admit singular minimizers or may exhibit Lavrentiev phenomenon, when minimizers in different functions spaces are different, and in many cases it is not known whether the weak formulation of the Euler-Lagrange equations makes sense [3].

The present work is devoted to validation of variational principle for spatial quasi-isometric mappings. In [5] it was suggested a polyconvex functional defined on the subset of mappings with bounded distortion in the sense of [4] and with bounded metric distortion.

The existence theorem for this functional is proved here under natural assumptions on the boundary of domain (Lipschits-continuity). We show that functional takes finite values only on quasi-isometric mappings, and any minimizer is quasi-isometric mapping as well. The minimal regularity of solutions is restricted to locally-Lipschits mappings meaning that the Lavrentiev phenomenon does not take place. In [4] it was shown that any mapping with bounded distortion can be obtained as a minimizer of the Dirichlet-type integral. We obtain similar result, namely every quasi-isometric mapping can be obtained as the minimizer of the functional [5], meaning that the suggested variational description of quasi-isometric mappings is essentially complete.

The regularity of solutions in principle allows to show the convergence of minimizers in the class of piecewise-affine mappings (such as those arising in finite elements methods) to the exact minimizers.

The theoretical analysis supports numerical evidences which show that mapping construction technique based on suggested functional is robust, stable and easy controllable.

Nevertheless there are still several open theoretical problems. The most important one is the stability of minimizers and conditions for global uniqueness of solutions.

References

[1] S.K. Godunov, V.M. Gordienko and G.A. Chumakov. Quasi-isometric parameterization of a curvilinear quadrangle and a metric of constant curvature, *Siberian Advances in Mathematics* 1995, **5**(2):1–20.

[2] J.M. Ball. Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* 1977, **63**:337–403.

[3] J.M. Ball. Singularities and computation of minimizers for variational problems, 1999. Preprint of Math. Institute of Univ. of Oxford.

[4] Y.G. Reshetnyak. Mappings with bounded deformation as extremals of Dirichlet type integrals. *Siberian Math. J.* 1968, **9**:487–498.

[5] V.A. Garanzha. Barrier construction of quasi-isometric grids. *Comp. Math. Math. Phys.* 2000, **40**:1617–1637.