Metric control of spatial mappings

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Problems of interest in this work are related to optimal grid generation, adaptation, surface parameterization, surface flattening and morphing with guaranteed quality.

Basic requirements for mapping construction methods in the above applications can be formulated as follows: max-norm control of local shape and volume distortion, orientation control, smoothness and stability. None of methods described in literature can do above with rigorous foundations available/applicable for practically interesting cases.

Short and obviously incomplete list of real-life applications of surface flattening or mapping of one surface onto another is presented below:
- optimal parameterization and surface reconstruction;
- sheet metal forming problems, multipass stamping;
- flattening of faults for reconstruction of geological structures;
- texture mappings and morphing in computer graphics.
- brain mapping: construction of canonical mappings; flat cortical mapping, mapping of the surface of brain on the sphere;
- virtual endoscopy and colonoscopy, and other biomedical applications - flattening and unfolding of colon-like structures; mapping on the surface of cylinder;

Denote the spatial mapping to be constructed by $x(\xi)$. For construction of controlled mappings we consider minimization of the polyconvex functional suggested in [3]. This variational problem is well-posed [5]. In terms of practical means for controlling the mappings one should consider several important cases:

1) functional $\int g(\nabla x) d\xi$ which describes "quasi-uniform" mappings, where $g(\cdot)$ depends on orthogonal invariants of matrix $\nabla x^T \nabla x$.

2) Control via prescribed metrics in logical space $H : h_{ij}(\xi)$ or via metrics in "physical coordinates" $G : g_{ij}(x)$, where in general case g is the function of invariants of matrix $\nabla x^T G \nabla x H^{-1}$.

Case H: $\int g(\nabla x, \xi) d\xi$ covers local control of mapping properties - shape, size and orientation control, apriori stretching, orthogonality near boundary. Low regularity of $g(\cdot, \xi)$ is acceptable, in particular it is not supposed to be continuous. Numerical methods for minimization of discrete functional behave quite good in this case.

Case G: $\int g(\nabla x, x) d\xi$, covers adaptation, surface parameterization, etc. Low regularity of $g(., x)$ is highly undesirable from theoretical point of view and naturally led to many numerical problems. In practice in many applications metrics $G(x)$ depends on the gradient of a monitoring (vector-valued) function $f(x)$. When $f(x)$ is piecewise smooth then $G(x)$ is not continuous! In this case the approximation of functional becomes a problem since special care should be taken in order to compute terms depending on $G(x)$. Discrete minimization encounters lots of problems. The gradient of functional depends on the second derivatives of $f(x)$ and locally may become quite meaningless. So in practice the only mapped grid methods which are stable enough are those where the approximate gradient of functional is computed with frozen $G(x)$. Anyway lack of convergence of minimization procedure in this case still may be present.

We suggest that the solution to this problem should be based on the A.D. Alexandrov theory of manyfolds with bounded curvature. Very loosely speaking, Alexandrov class of surfaces with bounded curvature consists of the surfaces being the limits of a convergent sequence of polyhedral surfaces with uniformly bounded positive part of curvature. The curvature of polyhedral surfaces is computed using the angle defect around each vertex. Basic operation in the Alexandrov theory is a flattening of geodesic triangle preserving the length of edges. In terms of grid optimization this operation means that given three vertices of triangle x_1, x_2, x_3, we compute geodesic triangle based on metrics $G(x)$ and flatten it. The deviation of resulting flat triangle from prescribed shape allows to compute the deformation metrics for each triangle which should compensate for wrong shape. From the algebraic point of view we approximate the problem with metrics $G(x)$ by a convergent sequence of problems with $H(\xi)$.

Important test case of this approach is related to parameterization of facetized surfaces. The quasi-isometric flattening procedure by itself provides a good enough parameterization and in most cases there is no need for the parameterization on the plane to take into account residual distortion [6] which also can be presented as a metrics.

However, when large facets are present and surface is far from developable then the piecewise-affine model of surface becomes overly stiff and the traces of nonsmoothness can be present in the final parameterization. The above described procedure allowed to eliminate them completely, while straighforward application of other approximation methods using residual distortion metrics was not successful.

As other test cases we consider surface reconstruction, surface grid generation and shape design for automotive applications as well as surface reconstruction and computation of canonical mappings for biomedical problems.

References

[1] S.K. Godunov, V.M. Gordienko and G.A. Chumakov. Quasi-isometric parameterization of a curvilinear quadrangle and a metric of constant curvature, *Siberian Advances in Mathematics* 1995, **5**(2):1–20.

[2] J.M. Ball. Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* 1977, **63**:337–403.

[3] V.A. Garanzha. Barrier construction of quasi-isometric grids. *Comp. Math. Math. Phys.* 2000, **40**:1617–1637.

[4] Reshetnyak, Yu: Two-Dimensional Manifolds of Bounded Curvature. In: Reshetnyak Yu.(ed.), Geometry IV (Non-regular Riemannian Geometry), pp. 3-165. Springer-Verlag, Berlin (1991).

[5] V.A. Garanzha and N.L. Zamarashkin. Spatial quasi-isometric mappings as minimizers of polyconvex functionals, paper to be presented at the workshop “Grid generation: theory and applications”, Moscow 2002.

[6] V.A. Garanzha. Max-norm optimization of spatial mappings with application to grid generation, construction of surfaces and shape design. “Grid generation: new trends and applications in real-world simulations”, eds. S.A. Ivanenko, V.A. Garanzha, Comm. on Appl. Math., Computer Centre, Russian Academy of Sciences, Moscow, 2001