Curvature criteria in surface reconstruction

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In this communication we discuss the problem of reconstructing surfaces from discrete scattered (irregularly distributed) data. One of the usual ways to preprocess the data is to construct first an initial triangulated polyhedral surface that spans all given data points. The data points will then be the vertices of the obtained triangulated surface.
We refer to the surface, which we want to reconstruct, as to the *source surface*. A polyhedral (triangulated) surface, reconstructed from the given discrete data set, can be considered as a *polyhedral model* of the source surface.

The key question is:

- How to determine the measure of adequacy of a polyhedral model with respect to the source surface?

A surface posseses *topological* and *metric* properties. The metric properties are those that depend on the extent and details of the shape of the features in the structure under study. The notion of the shape (not well–defined) is directly related to the concept of curvature (well–defined). There exist several types of curvature for a surface, the main ones are the Gaussian and the Mean curvatures. Topology concerns with those properties of the shape that do not change under deformation (one-to-one and bicontinuous).
In this communication we assume that possible polyhedral models preserve the main topological characteristics of the source surface, *i.e.*, its *genus* and *orientability*. Therefore, if we could estimate the curvatures (up to an admissible tolerance) of the source surface from a polyhedral model, then we could consider this model as adequate.
If we know the analytical representation of the surface, then we can easily compute its curvatures by using the well–known formulae of differential geometry [5]. These formulae require high order differentiability of the surface (at least the second order) and, therefore, are not applicable to polyhedral surfaces, which are of C^0–class. Nevertheless, analogues of curvatures for polyhedral surfaces exist and easily can be computed [1, 4]. But the number of polyhedral models is huge, and many of them might be quite inadequate.
We investigate which polyhedral model best approximates the curvatures of the source surface. We can pass from one polyhedral model to another just by swapping an edge [2]. This operation can be considered as an optimisation procedure, and, hence, our problem to find an adequate polyhedral model can be transformed into the problem of optimisation of an initial polyhedral model.

*The research of the author is supported by the NWO (STW) (Dutch Organisation for Scientific Research), project No. TWI4816

We give a brief overview of the theory of non-regular surfaces developed by A.D. Aleksandrov and his school in Russia [6]. Polyhedral surfaces, as *polyhedral metrics*, form the basic part of this theory. Then we discuss several geometric variational criteria to obtain an adequate polyhedral model. All these criteria are related to minimisation of one or another curvature, defined for a polyhedral surface. Most of these criteria have been introduced by the author and her long-term collaborator Ruud van Damme [1, 2].

The last topic which we discuss is the relation between the data and the adequacy of the model. Our model can be optimal among all polyhedral models with respect to the given criterion, but still be non–adequate, because the given data do not sufficiently represent the source object. Therefore, the second key question is:

- How to determine whether the data are sufficient?

We discuss an approach to answer this question, by using some notions of the critical point theory and the theory of non–regular surfaces [3, 6].

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