Optimality principle for nondegenerate grids [1]

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Boundary-conforming invertible map of a unit square onto a physical domain generates unfolded grid. If a map is not invertible, it produces a folded grid, which cannot be used in computations. A variational principle is formulated for the set of all invertible maps, being the extensions of a given boundary map. The principle allows to separate this set from all other maps.

The functional, called the energy density of a map is considered. It depends on first derivatives of coordinates and variable coefficients. These coefficients form a symmetric positive-definite matrix. The variational principle states that a map will be invertible if and only if it is a minimum of the energy density functional. From this it follows that the Euler equations of the functional describe all possible invertible (smooth and one-to-one) maps.

A discrete analog of the variational principle, based on the energy density of grid deformation is formulated. Complete proof of the principle in this case is presented.

The energy density of grid deformation is a discrete functional with an infinite barrier at the boundary of the set of unfolded grids. The barrier property is very important in problems with moving boundaries and in moving adaptive grid technology because such methods ensure generation of unfolded grids at every time step. Direct control of cell shapes is used to provide grid orthogonality with prescribed width of cells near the boundary. Adaptation to the solution of host equations is realized as in the adaptive-harmonic grid generator.

Variational principle is implemented in the method for the numerical solution of shallow water equations. Simulation of wind-induced currents in the Mojaiskii reservoir has been performed and results are presented. Adaptation to the numerical solution with boundary orthogonality allows to increase the accuracy of computations, which is consistent with experimental data.

Keywords: invertible map, unfolded grid, equidistribution principle, minimization problem

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Introduction

An equidistribution principle has been developed for grid generation in the one-dimensional case (see [1]). A positive function multiplied by the Jacobian of the transformation is set to be equal to a positive constant

$$\omega x_\xi = const > 0, \quad \omega(\xi) > 0, \quad \xi \in (0,1), \tag{1a}$$

with boundary conditions

$$x(0) = 0, \quad x(1) = 1. \tag{1b}$$

Instead of (1) the minimization problem can be considered for the functional

$$F = \int_0^1 \omega x_\xi^2 d\xi$$

with the Euler equation:

$$(\omega x_\xi)_\xi = 0.$$

Almost all adaptive grid generation methods in 1D are based on the equidistribution principle (1). This follows from the fact that if a smooth function $x(\xi)$ satisfies (1), then it will be an invertible mapping, since from (1) it follows that the Jacobian is strictly positive. On the other hand, for any invertible mapping $x^*(\xi)$ with positive Jacobian a positive weight function can be introduced as $\omega = const(x_\xi^*)^{-1}$. As a result $x^*(\xi)$ will satisfy (1). Consequently, both a variational formulation and the Euler equation contain all possible invertible mappings as their solutions.

Simplicity and clear geometric meaning of the equidistribution principle have led to the existence of many attempts to extend the principle to the multidimensional case (see [2], [3], [4]).

As one of possible approaches the following system of equations can be considered in the two-dimensional case

$$(\omega_1 x_\xi)_\xi + (\omega_2 x_\eta)_\eta = 0, \quad (\omega_1 y_\xi)_\xi + (\omega_2 y_\eta)_\eta = 0.$$

However, the class of all solutions of these equations for positive ω_1 and ω_2 contains functions $x(\xi,\eta)$, $y(\xi,\eta)$, which cannot ensure one-to-one mapping of a square onto a domain Ω. This is also valid for many other differential and variational statements that are used now for grid generation.

If a mapping is sought as a composition of algebraic map and inverse to harmonic [5], [6], then the composite mapping will be invertible. However,

the class of all solutions of the corresponding equations does not contain all possible invertible mappings.

At the present paper a variational principle which is a direct extension of the equidistribution principle to the 2D case is described. It is formulated both at continuous and discrete levels. The complete proof of the principle is presented for a discrete mapping, i.e. a structured grid.

1 Formulation of the variational principle

Let a smooth one-to-one mapping of the boundary of the unit square $Q = \{(\xi,\eta): \quad 0<\xi<1,\ 0<\eta<1\}$ onto the boundary of a domain Ω be given as $(x_\Gamma, y_\Gamma): \partial Q \to \partial\Omega$. Consider a problem of constructing a smooth extension of the functions x_Γ, y_Γ to the interior of the square, such that the resulting mapping $x(\xi,\eta)$, $y(\xi,\eta)$ will be invertible.

Consider the energy density functional (see also [7], [8])

$$F = \frac{1}{2}\int\int \frac{(x_\xi^2+y_\xi^2)G_{22} - 2(x_\xi x_\eta + y_\xi y_\eta)G_{12} + (x_\eta^2+y_\eta^2)G_{11}}{(x_\xi y_\eta - x_\eta y_\xi)\sqrt{G_{11}G_{22}-G_{12}^2}} d\xi d\eta. \quad (2)$$

Here $\{G_{lm},\ l,m=1,2\}$ are the elements of symmetric and positive-definite matrix $G(\xi,\eta) = G^T > 0$, defined in the interior of the unit square Q. Functional (2) is minimized at the class of functions $x(\xi,\eta)$, $y(\xi,\eta)$ being an extension of the fuctions $(x_\Gamma, y_\Gamma): \partial Q \to \partial\Omega$, defining a smooth invertible mapping between boundaries, to the interior of a square.

Consider simmetric and positive-definite matrix $g = (g_{lm})$ with elements

$$g_{11} = x_\xi^2 + y_\xi^2, \quad g_{12} = g_{21} = x_\xi x_\eta + y_\xi y_\eta, \quad g_{22} = x_\eta^2 + y_\eta^2. \quad (3)$$

It is easy to see that the integrand in (2) can be written as

$$\frac{1}{2}\frac{tr(G^{-1}g)}{\sqrt{det(G^{-1}g)}} = \frac{1}{2}\frac{\lambda_1+\lambda_2}{\sqrt{\lambda_1\lambda_2}} \geq 1, \quad (4)$$

where λ_1 and λ_2 are the eigenvalues of the matrix $G^{-1}g$. Equality in (4) is attained if and only if $\lambda_1 = \lambda_2$. From this it follows that $F \geq 1$, and $F = 1$ if and only if

$$G_{lm} = \alpha(\xi,\eta) g_{lm}, \quad (5)$$

where $\alpha(\xi,\eta) > 0$ is an arbitrary smooth function. Consequently, for each invertible mapping with positive Jacobian there exists such a matrix

$G(\xi,\eta) = G^T > 0$, with elements defined by formulas (5), that the minimum of the functional (2) will be attained at this mapping.

Instead of G it is convenient to introduce the matrix $\tilde{G}$ with the elements $\tilde{G}_{lm} = G_{lm}/\sqrt{det(G)}$ such, that $det(\tilde{G}) = 1$. Then the functional (2) will take the form

$$F = \frac{1}{2}\int_0^1\int_0^1 \frac{(x_\xi^2 + y_\xi^2)\tilde{G}_{22} - 2(x_\xi x_\eta + y_\xi y_\eta)\tilde{G}_{12} + (x_\eta^2 + y_\eta^2)\tilde{G}_{11}}{x_\xi y_\eta - x_\eta y_\xi} d\xi d\eta. \tag{6}$$

The Euler equations of the functional (6) can be written as

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = x_\xi(-P\tilde{G}_{22} + Q\tilde{G}_{12}) + x_\eta(P\tilde{G}_{12} - Q\tilde{G}_{11}),$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = y_\xi(-P\tilde{G}_{22} + Q\tilde{G}_{12}) + y_\eta(P\tilde{G}_{12} - Q\tilde{G}_{11}),$$

where

$$\alpha = x_\eta^2 + y_\eta^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma = x_\xi^2 + y_\xi^2,$$

$$P = \frac{1}{2}\left(\alpha\frac{\partial\tilde{G}_{11}}{\partial\xi} - 2\beta\frac{\partial\tilde{G}_{12}}{\partial\xi} + \gamma\frac{\partial\tilde{G}_{22}}{\partial\xi}\right),$$

$$Q = \frac{1}{2}\left(\alpha\frac{\partial\tilde{G}_{11}}{\partial\eta} - 2\beta\frac{\partial\tilde{G}_{12}}{\partial\eta} + \gamma\frac{\partial\tilde{G}_{22}}{\partial\eta}\right).$$

Consider the class of all invertible mappings of a unit square Q onto a domain Ω, satisfying the same boundary conditions $(x_\Gamma, y_\Gamma) : \partial Q \to \partial\Omega$.

Let us formulate a conjecture, which is a variational principle allowing to separate the set of all invertible maps of a unit square onto a domain Ω from those maps that are not invertible.

Optimality principle. *A smooth mapping of a unit square Q onto a domain Ω is invertible if and only if it ensures the minimum of the functional F with some symmetric and positive-definite matrix G.*

2 Discrete case

The problem of grid generation in two dimensions will be considered in the following formulation. In a simply connected domain Ω on the plane x, y a grid

$$(x,y)_{ij},\ i = 1,\dots,i^*;\ j = 1,\dots,j^*,$$

must be constructed with given coordinates of boundary nodes

$$(x,y)_{i1},\ (x,y)_{ij^*},\ (x,y)_{1j},\ (x,y)_{i^*j}.$$

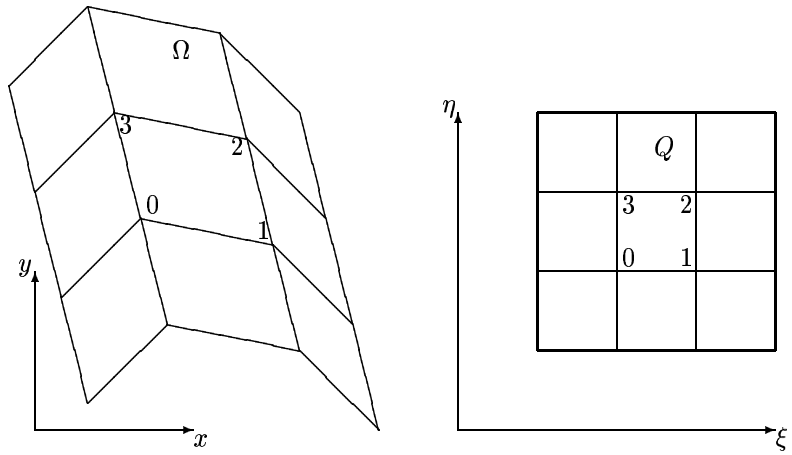

Figure 1: Correspondence between node numbers for a mapping of a square cell in the plane ξ, η onto the corresponding cell of the grid in the plane x, y

The problem can be treated as a discrete analog of the problem of finding functions $x(\xi, \eta)$ and $y(\xi, \eta)$, ensuring one-to-one mapping of the parametric square onto a domain Ω (see fig.1) with a given transformation between boundaries $(x_\Gamma, y_\Gamma) : \partial Q \to \partial\Omega$.

Let the coordinates $(x, y)_{i,j}$ of grid nodes be given. To construct the mapping $x^h(\xi, \eta)$, $y^h(\xi, \eta)$ of the parametric square onto the domain Ω such that $x^h(i, j) = x_{i,j}$ and $y^h(i, j) = y_{ij}$ we use quadrilateral isoparametric finite elements. The square cell numbered $i+1/2, j+1/2$ in the plane ξ, η is mapped onto the quadrilateral cell in the plane x, y , formed by nodes with coordinates $(x, y)_{i,j}$, $(x, y)_{i,j+1}$, $(x, y)_{i+1,j+1}$, $(x, y)_{i+1,j}$. The cell vertices are numbered from 0 to 3 in the counterclockwise direction as shown in fig.1. The node (i, j) corresponds to the vertex 0, node $(i, j + 1)$ to the vertex 1 and so on. Each vertex is associated with a triangle: vertex 0 with $\triangle_{301}$, vertex 1 with $\triangle_{012}$ and so on. The doubled area J_k, $k = 0, 1, 2, 3$, of these triangles is introduced as follows $J_0 = (x_1 - x_0)(y_3 - y_0) - (y_1 - y_0)(x_3 - x_0)$.

Discrete analog of the Jacobian positiveness can be written as a system of inequalities [9]

$$[J_k]_{i+1/2,j+1/2} > 0, \; k = 0, 1, 2, 3; \; i = 1, \ldots, i^* - 1; \; j = 1, \ldots, j^* - 1, \quad (7)$$

where $J_k = (x_{k+1} - x_k)(y_{k-1} - y_k) - (y_{k+1} - y_k)(x_{k-1} - x_k)$, index k is ciclic.

If conditions (7) are satisfied, then all grid cells are convex quadrilaterals.

The set of grids satisfying inequalities (7) is called the class of unfolded grids and denoted by W_D. This class is a subset of the Euclidean space R^N, where $N = 2(i^* - 2)(j^* - 2)$ is the total number of degrees of freedom of the grid equal to double the number of its internal nodes. In this space W_D is an open bounded set. Its boundary ∂W_D is the set of grids for which at least one of the inequalities (7) becomes an equality.

Every admissible set of boundary nodes defines a corresponding class W_D. In order to characterize each of these classes it is natural to apply discrete analog of the variational principle from the previous section.

Discrete analog of the functional (6) can be written in the form

$$F^h = \frac{1}{(i^*-1)(j^*-1)} \sum_{i=1}^{i^*-1} \sum_{j=1}^{j^*-1} \sum_{k=0}^{3} \frac{1}{4} [F_k]_{i+1/2,j+1/2}\,, \tag{8a}$$

$$F_k = \frac{(r_{k+1}-r_k)^2(\tilde{G}_{22})_k - 2(r_{k+1}-r_k)(r_{k-1}-r_k)(\tilde{G}_{12})_k + (r_{k-1}-r_k)^2(\tilde{G}_{11})_k}{2[(x_{k+1}-x_k)(y_{k-1}-y_k) - (y_{k+1}-y_k)(x_{k-1}-x_k)]}, \tag{8b}$$

where

$$r_k = (x_k, y_k)^T,\ (r_{k+1}-r_k)^2 = (x_{k+1}-x_k)^2 + (y_{k+1}-y_k)^2,$$

$$(r_{k+1}-r_k)(r_{k-1}-r_k) = (x_{k+1}-x_k)(x_{k-1}-x_k) + (y_{k+1}-y_k)(y_{k-1}-y_k),$$

$$(r_{k-1}-r_k)^2 = (x_{k-1}-x_k)^2 + (y_{k-1}-y_k)^2,$$

$(\tilde{G}_{lm})_k$ are elements of the matrix $\tilde{G}_k = \tilde{G}_k^T > 0,\ \ det(\tilde{G}_k) = 1$, refered to a tringle with the local number k of a cell $i+1/2, j+1/2$.

Consider the set of symmetric and positive-definite matrices g_k with the elements

$$(g_{11})_k = (r_{k+1}-r_k)^2,\ (g_{12})_k = (g_{21})_k = (r_{k+1}-r_k)(r_{k-1}-r_k),$$

$$(g_{22})_k = (r_{k-1}-r_k)^2. \tag{9}$$

Substituting (9) into (8b) we obtain

$$F_k = \frac{tr(\tilde{G}_k^{-1} g_k)}{2\sqrt{det(\tilde{G}_k^{-1} g_k)}} = \frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1 \lambda_2}} \geq 1, \tag{10}$$

where λ_1 and λ_2 are the eigenvalues of the matrix $\tilde{G}_k^{-1} g_k$. The strict equality in (10) is attained if and only if $\lambda_1 = \lambda_2$. From this it follows that $F_k \geq 1$, and $F_k = 1$ if and only if

$$(\tilde{G}_{lm})_k = (g_{lm})_k / \sqrt{det(g_k)}. \tag{11}$$

The function F^h possesses the so-called barrier property, i.e. the following Lemma holds.

Lemma. *For any set of symmetric and positive-definite matrices defined at every cell* $\{\tilde{G}_k(ij),\ k = 0,1,2,3;\ i = 1,\ldots,i^*-1;\ j = 1,\ldots,j^*-1\}$ *such that* $det(\tilde{G}_k) = 1$, *the function* F^h *has an infinite barrier at the boundary of the class of unfolded grids* W_D.

Proof. First, one can notice that for the proof of the Lemma it is enough to show that if for some cell the area of at least one of triangles tends to zero, while remaining positive, but the length of at least one side does not tend to zero, then $F^h \to +\infty$. Indeed, if for some cell there exists such k, that $J_k \to 0$ but F^h does not tend to $+\infty$, then the numerator in (8b) must also tend to zero. From this and from the inequality

$$(r_{k+1} - r_k)^2 (\tilde{G}_{22})_k - 2(r_{k+1} - r_k)(r_{k-1} - r_k)(\tilde{G}_{12})_k + (r_{k-1} - r_k)^2 (\tilde{G}_{11})_k$$
$$\geq \lambda_{min}(\tilde{G}_k) \left[(r_{k+1} - r_k)^2 + (r_{k-1} - r_k)^2 \right]$$

it follows that lengths of two sides of the cell tend to zero. Consequently, the areas of two triangles that contain these sides must also tend to zero. Repeating the previous argumentation we obtain that the lengths of all sides of the cell must tend to zero. Obtained contradiction proves the Lemma. □

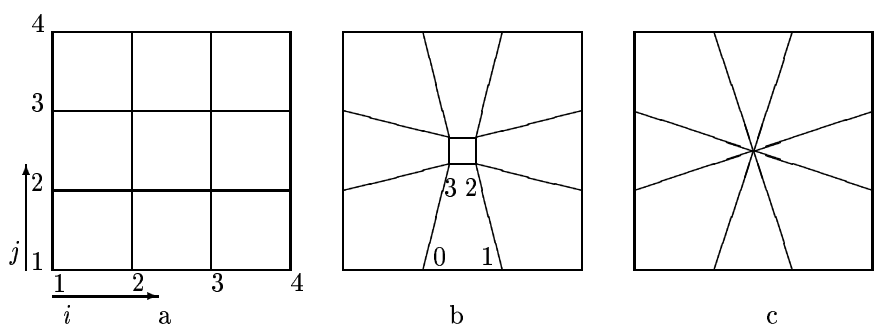

Figure 2: Grid deformation such that the central cell degenerates into a point

To illustrate the Lemma consider the process of 4x4 grid deformation, shown in fig.2. Initial grid is shown in fig.2a, the final grid is shown in fig.2c. All sides of the central cell tend to zero and, consequently, areas of all triangles also tend to zero. One can see that for the cell (2+1/2,1+1/2) with local enumeration shown in fig.2b, the area of a triangle S_{023} tends to zero. At the same time the length l_{03} does not tend to zero. From this it follows that the numerator in the corresponding term $[F_3]_{2+1/2,1+1/2}$ of F^h (8) does not tend to zero. Consequently, F^h tends to infinity.

3 Optimality principle at the discrete level

In this section the proof of a discrete counterpart of the optimality principle will be given.

Theorem. *A structured grid, constructed in a domain Ω for the given coordinates of boundary nodes is an element of the class of unfolded grids W_D if and only if it is a minimum of the functional F^h for some set of symmetric and positive-definite matrices $\tilde{G}_k(ij)$ such, that $det(\tilde{G}_k) = 1$.*

Proof of the necessity. Let a grid satisfying inequalities (7) be given. Let in each cell four matrices $\tilde{G}_k,\ k = 0, 1, 2, 3$ be introduces by formulas (9), (11). It is easy to see that in this case each term in (8a) is equal to 1:

$$F_k = \frac{tr(\tilde{G}_k^{-1} g_k)}{2\sqrt{\tilde{G}_k^{-1} det(g_k)}} = \frac{tr(g_k^{-1} g_k)}{2\sqrt{det(g_k^{-1} g_k)}} = 1.$$

Consequently, for every grid from W_D there exists such a set of matrices $\tilde{G}_k(ij)$, that F^h takes a minimal possible value, equal to 1. Let us show that the grid, which corresponds to a minimum $F^h = 1$ is unique.

Indeed, if $F^h = 1$, then each term in (8a) takes the minimal possible value equal to $F_k = 1$. Considering the sum of four terms for one cell, we can see that if coordinates of any two adjacent vertices are specified, the last two are determined uniquelly from the conditions $F_k = 1, k = 0, 1, 2, 3$. Hence, if $F^h = 1$, the grid can be constructed sequentially, starting from the boundary. At the beginning for a given coordinates of two adjacent boundary points the coordinates of two other points of the cell are defined. Then the procedure is repeated for the neighbour cell and so on. As a result a grid will be uniquelly constructed from boundary points distribution.

Proof of the sufficiency. Let a set of symmetric and positive-definite matrices $\tilde{G}_k(ij)$ be given. The function F^h has an infinite barrier at the boundary of the class of unfolded grids W_D according to the Lemma on

barrier property. From (10) it follows that F^h is bounded below by 1. Since F^h as a fuction of internal node coordinates is continuous in W_D, there exists at least one grid in W_D such that it is the minimum of F^h. Moreover, F^h is a continuously differentiable function of internal node coordinates. From this follows that the system of algebraic equation (the necessary conditions of minimum) $\partial F^h/\partial x_{ij} = 0$, $\partial F^h/\partial y_{ij} = 0$, has at least one solution which is an unfolded grid from W_D. This completes the proof. □

Remark 1. Problem formulation may include block-structured grids. Optimality principle can be extended to general unstructured grids.

Remark 2. The variational principle can be extended to the 3D case of unstructured tetrahedral and hexahedral meshes.

4 Adaptation

The functional (2) can be extended to the case of adaptation [8]

$$F_a = \frac{1}{2}\int\int \frac{\alpha\tilde{g}_{11} + 2\beta\tilde{g}_{12} + \gamma\tilde{g}_{22}}{(x_\xi y_\eta - x_\eta y_\xi)\sqrt{\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2}\sqrt{G_{11}G_{22} - G_{12}^2}} d\xi d\eta, \qquad (12)$$

where

$$\alpha = x_\xi^2 G_{22} - 2x_\xi x_\eta G_{12} + x_\eta^2 G_{11},$$
$$\beta = x_\xi y_\xi G_{22} - (x_\xi y_\eta + x_\eta y_\xi)G_{12} + x_\eta y_\eta G_{11},$$
$$\gamma = y_\xi^2 G_{22} - 2y_\xi y_\eta G_{12} + y_\eta^2 G_{11}.$$

The functional F_a (12) contains metric coefficients $\tilde{g}_{ij}$ that can be used for adaptation to the solution of host equations. Elements of the matrix $\tilde{g}_{ij}$ can be computed as follows

$$\tilde{g}_{11} = 1 + \sum_i c_i^2 (f_i)_x^2, \quad \tilde{g}_{12} = \sum_i c_i^2 (f_i)_x (f_i)_y, \quad \tilde{g}_{22} = 1 + \sum_i c_i^2 (f_i)_y^2.$$

In this case the minimization problem for the functional (12) is equivalent to the problem of constructing such a curvilinear coordinate system on the monitor surface of vector-function

$$r(x, y) = (c_1 f_1, c_2 f_2 ...)(x, y),$$

that the mean energy density F_a for the mapping $(\xi, \eta) \to (x, y, r(x, y))$ is minimized. As a result, one obtains a surface parameterization of the form $(x, y, c_1 f_1, c_2 f_2 ...)(\xi, \eta)$, which corresponds to a structured grid. In the case of an unstructured grid, the problem is posed for the energy density for the

mapping of a given cell in the plane ξ, η with metric G to the corresponding surface cell.

Approximation and minimization of the functional (12) can be done by applying the method described in detail in [9].

5 Simulation of wind-induced circulation in the Mojaiskii reservoir

We consider the model of currents in shallow water body, described by shallow water equations

$$\frac{\partial u}{\partial t} - l_C v = -g\frac{\partial \varsigma}{\partial x} - \frac{r}{h}u + \frac{\tau_x}{h},$$
$$\frac{\partial v}{\partial t} + l_C u = -g\frac{\partial \varsigma}{\partial y} - \frac{r}{h}v + \frac{\tau_y}{h},$$
$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0.$$

Here u and v are the components of the averaged velocity vector, $\varsigma(x, y, t)$ is the free surface level, $h(x, y)$ is the depth, τ_x and τ_y are the components of the frictional stress along axes x y respectively, l_C is the Coriolis parameter, equal to $l_C = 2\omega \sin\theta$, where ω is the Earth's angular velocity, θ is the latitude, g is the acceleration due to gravity, $r = C_f\sqrt{u^2 + v^2}$, C_f is the bottom friction coefficient.

The condition on the boundary is

$$u_n = un_x + vn_y = 0,$$

where n_x and n_y are the components of the external normal to the boundary.

Introduce the integral stream function

$$hu = \frac{\partial \Psi}{\partial y}, \quad hv = -\frac{\partial \Psi}{\partial x}$$

and obtain an equation which we shall solve in the simply connected domain Ω with the boundary Γ.

$$\frac{\partial}{\partial t}[\frac{\partial}{\partial x}(\frac{1}{h}\frac{\partial \Psi}{\partial x}) + \frac{\partial}{\partial y}(\frac{1}{h}\frac{\partial \Psi}{\partial y})] + \frac{\partial}{\partial y}(\frac{l_C}{h}\frac{\partial \Psi}{\partial x}) -$$
$$\frac{\partial}{\partial x}(\frac{l_C}{h}\frac{\partial \Psi}{\partial y}) + \frac{\partial}{\partial x}(\frac{r}{h}\frac{\partial \Psi}{\partial x}) + \frac{\partial}{\partial y}(\frac{r}{h}\frac{\partial \Psi}{\partial y}) = \frac{\partial}{\partial y}(\frac{\tau_x}{h}) - \frac{\partial}{\partial x}(\frac{\tau_y}{h}).$$

The initial and boundary conditions are written in the form

$$\Psi \mid_{\Gamma} = 0, \quad \Psi \mid_{t=0} = \Psi_0(x, y).$$

We assign the components of the frictional stress of the wind by means of formula

$$\tau_x = C_W W_x \sqrt{W_x + W_y}, \quad \tau_y = C_W W_y \sqrt{W_x + W_y},$$

where W_x W_y are the components of the wind velocity, C_W is a coefficient.

The implicit scheme is used to approximate the equation for the stream function with respect to time. The resulting elliptic equation for the integral stream function from the upper time level is solved by the upstream finite element method described in detail in [10].

The input data were the values of the water body depth at nodes of a square grid and the wind velocity. These values were interpolated to the adaptive grid at each iteration using bilinear interpolation formulae.

The wind circulation was simulated with the following parameter walues: $C_f = 2.6 \cdot 10^{-3}$, $C_W = 3.1 \cdot 10^{-6}$, $g = 9.8\ m/s^2$. Results of computations performed for the western wind of speed 3 m/s are shown in figs.3-5.

In fig.3 results of computations for Mojaiskii reservoir obtained with the use of 147x53 harmonic grid are presented. Velosities are also shown in the fig.3. Grid does not condense toward the boundary and this leads to the lost of accuracy.

Result of computations obtained on the grid with the same number of nodes are shown in fig.4. The grid was orthogonalized and condensed toward the top and bottom boundaries. It was constructed with the use of the functional (2). Elements of the matrices G_{ij} were computed as described in [7]. Obtained grid much better describes the geometry and results of computations exhibit the better resolution of currents near the boundary.

In fig.5 the grid and velocities are shown in the case of adaptation to the numerical solution with grid clustering and orthogonalization toward the boundary. The functional (12) is used for adaptive grid generation. Adaptation to the numerical solution with boundary orthogonality allows to increase the accuracy of computations, which is consistent with experimental data.

Conclusions

In principle, the technique described in this paper can be applied to generate any grid of prescribed structure.This advantage should fully manifest itself when the technique is implemented in interactive software.

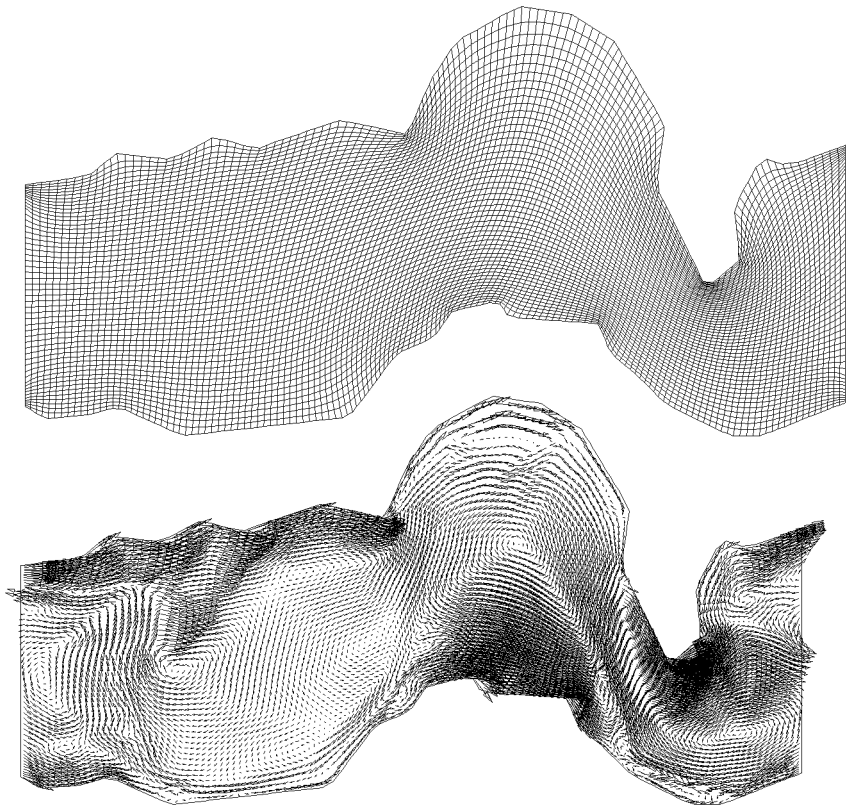

Figure 3: Harmonic 147x53 grid and velocities obtained on this grid

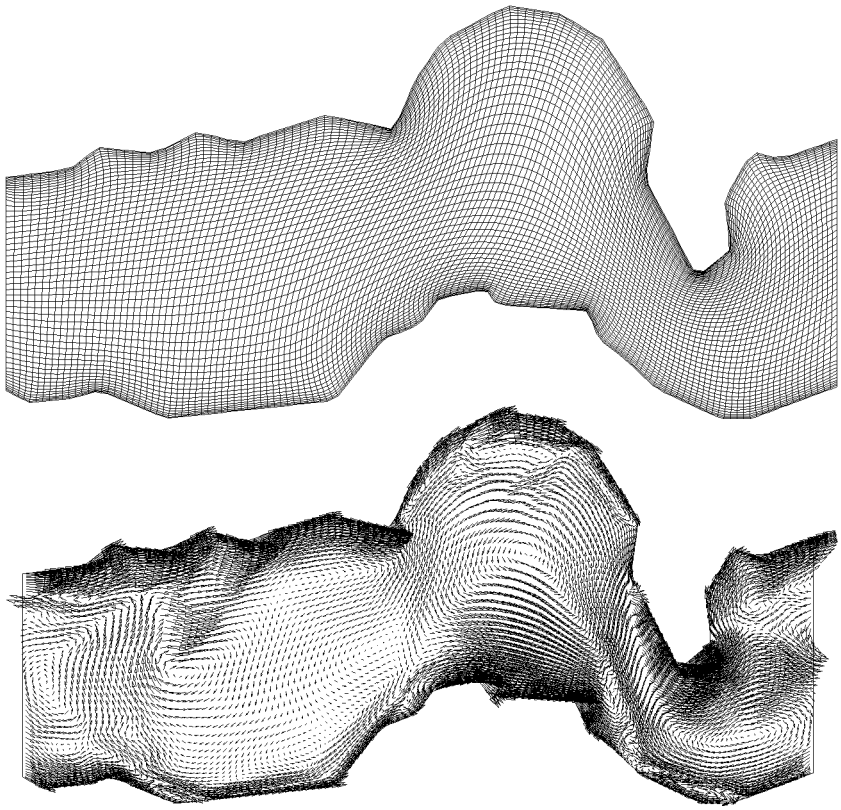

Figure 4: Results of computations without adaptation

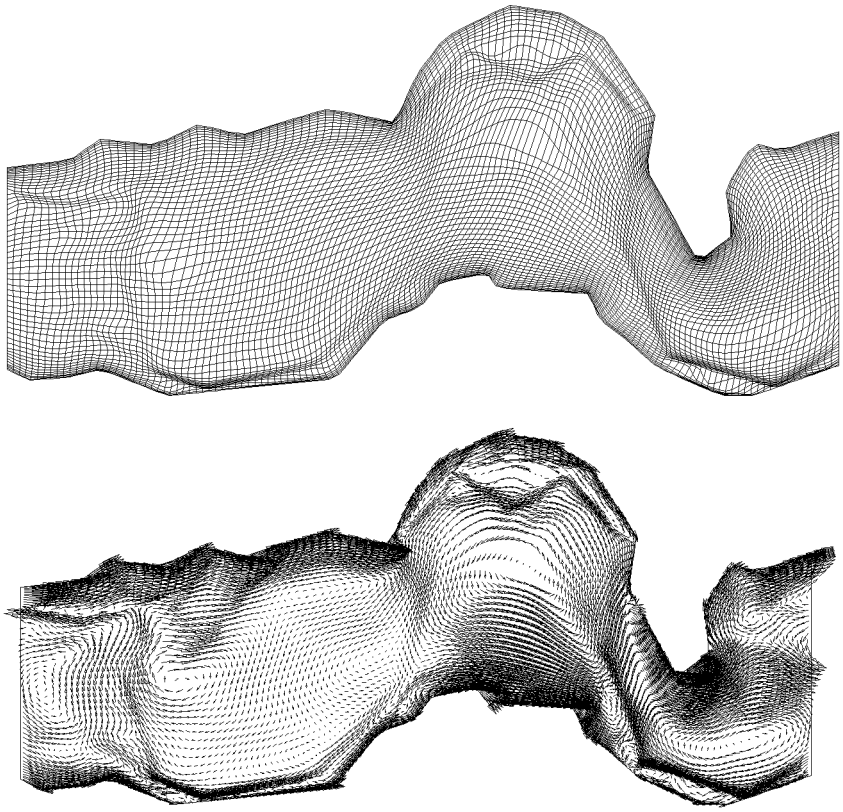

Figure 5: Results of computations with adaptation

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