# An Approach to Generation of Quasi–isometric Grids

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## 1 INTRODUCTION

A multiblock grid technology is widespread in fluid dynamics computations. Grid in a multiblock domain can be constructed by a mapping of each block onto the unit square. The grid in each block is regular but block structure can be irregular. In our approach the origin physical domain is subdivided into quadrangular blocks (sub-domains) by an iteration process of grid generation. It should be given the information of the topological structure of blocks and some geometric data such as position of outer boundaries of the original domain, size of grid, and, possibly position of some vertices of blocks, some angles of intersection of block boundaries. But this information is not sufficient to construct the multiblock configuration by means of a simple way. In this paper a method of block structured quadrangular grid generation on a plane, based on the theory of quasi-isometric mapping [1], is given; some aspects of the problem under consideration are discussed. Note, a lot of researches has focused on the grid generation based on mapping technique [2]. However inspite of achieved progress there is often no proof that the used technique provides a mapping of the needed properties.

In our approach the block interfaces and grid itself are constructed by discretization of quasi-isometric mapping of the union of the squares onto the original domain. It is significant that there is no necessity to specify the position of the block interfaces in the initial data. In this sense we can say about automation of the grid generation process.

The main emphasis of this paper is on the computational implementation of the quasi-isometric mapping technique. Section 2 describes the basic variational principle and the numerical method for constructing a quasi-isometric grid in a curvilinear quadrangle. Section 3 presents the generalization of this approach for a multiblock domain.

# 2 SINGLE REGION FORMULATION

## 2.1 BASIC PRINCIPLE

We describe the quasi-isometric mapping technique for a curvilinear quadrangle $\Omega \subset R^2$ with four sides and the angles $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ (Fig. 1). Let us mention some definitions and results from [1].

A curvilinear quadrangle $\Omega$ is called smooth if each of its sides is sufficiently smooth (Lyapunov arc with Hölder exponent $\mu$). A mapping $u = u(s,t), v = v(s,t)$ of the unit square $K_0 = \{(s,t) : 0 \leq s,t \leq 1\}$ onto $\Omega$ is called quasi-isometric if the ration between two (sufficiently close) points $(s_1, t_1)$ and $(s_2, t_2)$ to the distance between their images $(u_1, v_1)$ and $(u_2, v_2)$ is bounded:

$$0 < \sigma_1 \leq \frac{\sqrt{(u_2 - u_1)^2 + (v_2 - v_1)^2}}{\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2}} \leq \sigma_2.$$

A mapping $u(s,t), v(s,t)$ of the domain $\Omega$ is called $C^1_\mu(\Omega)$-mapping if the partial derivatives $u_s, u_t, v_s, v_t$ are continuous and satisfy the Hölder condition with exponent $\mu$. We can also say that a mapping is called quasi-isometric if $|u_s|, |u_t|, |v_s|, |v_t| \leq H$, $u_s v_t - u_t v_s \geq h > 0$.

Note that a quasi-isometric mapping is a quasi-conformal one, i.e. being conformal with respect to some metric. A conformal mapping may not be quasi-isometric. A mapping of a smooth quadrangle onto another one is conformal and quasi-isometric if and only if the corresponding angles and conformal modules of these quadrangles are equal [3]. The main result of [1] is following:

**Theorem 1.** Let $\Omega \subset R^2$ be a smooth quadrangle with boundary $\partial\Omega$ and angles $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ such that $0 < \varphi_j < \pi$ , $\delta \equiv \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 2\pi < 2\varphi_j$, $(j = 1, 2, 3, 4)$. Then any quasi-isometric $C^1_\mu$-mapping of the boundary $\partial K_0$ of the unit square $K_0$ onto $\partial\Omega$ can be extended to quasi-isometric $C^1_\mu(K_0)$-mapping of $K_0$ onto $\Omega$.

Quality of a quasi-isometric mapping is defined by the ratio $\sigma_1/\sigma_2$; a mapping is optimal if this ratio is maximal among all quasi-isometric mappings. There is a hypothesis that the mapping [1] is close to optimal one under the assumption that curvature of any image of a straight line segment is bounded.

The considered quasi-isometric mapping $u = u(s,t), v = v(s,t)$ of $K_0$ onto $\Omega$ is constructed as the superposition of two mappings (see Fig. 1): $u = u(x,y), v = v(x,y)$ maps the unit square $K = \{(x,y) : 0 \le x, y \le 1\}$ onto $\Omega$; $x = x(s,t), y = y(s,t)$ maps $K_0$ onto $K$. The first mapping $u, v$ is quasi-isometric and conformal with respect to the metric $g_{ij}$ chosen from a five-parameter family of metrics defined in the unit square $K$ by:

$$\begin{aligned}
&g_{11}(y) = 1 + k + (s_1 + s_2 - s_3 - s_4)\bar{y} - (s_1 + s_2 + s_3 + s_4)\bar{y}^2,\\
&g_{22}(x) = 1 - k + (s_1 - s_2 - s_3 + s_4)\bar{x} - (s_1 + s_2 + s_3 + s_4)\bar{x}^2,\\
&g_{12}(x,y) = \frac{1}{4}(s_1 - s_2 + s_3 - s_4) - \frac{1}{2}(s_1 + s_2 - s_3 - s_4)\bar{x} -\\
&-\frac{1}{2}(s_1 - s_2 - s_3 + s_4)\bar{y} + (s_1 + s_2 + s_3 + s_4)\bar{x}\bar{y},
\end{aligned} \tag{1}$$

(here $\bar{x} = x - \frac{1}{2}, \bar{y} = y - \frac{1}{2}$), and $g_{ij}$ are to satisfy the following relations in the verteces of $K$:

$$\begin{gathered}
s_i = \cos\varphi_i \sqrt{g_{11}(x_i, y_i) g_{22}(x_i, y_i)}, \quad i = \overline{1,4}\\
(x_1, y_1) = (0,0), (x_2, y_2) = (1,0), (x_3, y_3) = (1,1), (x_4, y_4) = (0,1).
\end{gathered} \tag{2}$$

The metrics (1) depends on five parameters $\vec{\mathbf{s}} = (s_1, s_2, s_3, s_4, k)$. The desired metric $g_{ij}$ represents the natural metric on the constant curvature surface (the Lobachevckian or Euclidean sphere or plane) such that geodesics in $K$ are straight line segments; the general form of such metrics is described in [4].

The mapping $u(x,y), v(x,y)$ can be constructed by the minimization of the functional

$$F = \frac{1}{2} \iint\limits_K \frac{\{g_{22}(u_x^2 + v_x^2) - 2g_{12}(u_x u_y + v_x v_y) + g_{11}(u_y^2 + v_y^2)\}}{\sqrt{g_{11}g_{22} - g_{12}^2}} dxdy, \tag{3}$$

with respect to $u, v$ and the metric parameters $\vec{s}$ under the conditions (2) and the free boundary condition.

The second mapping $x = x(s,t), y = y(s,t)$ of $K_0$ onto K is defined by four functions $x_0(s)$, $x_1(s)$, $y_0(t)$, $y_1(t)$, $0 \le s \le 1$, $0 \le t \le 1$, $x_i(0) = y_i(0) = 0$, $x_i(1) = y_i(1) = 1$, $i = 1, 2$. The mapping of the lower and upper sides of the square $K_0$ are defined by the functions $x_0(s)$ and $x_1(s)$, and the mapping of the left and right sides are defined by the functions $y_0(t)$ and $y_1(t)$. The functions $x_0, x_1, y_0, y_1$ are called the control functions, they are assumed to be smooth ($C^1_\mu$), strictly monotonically increasing, and their derivatives are bounded:

$$0 < \delta \le x_0'(s), x_1'(s), y_0'(t), y_1'(t) \le \Delta, (\delta \le 1).$$

The mapping of $K_0$ onto $K$ is the algebraic mapping given by the formulas:

$$\begin{aligned} x &= \omega^{-1}\{[1 - y_0(t)]x_0(s) + y_0(t)x_1(s)\}, \\ y &= \omega^{-1}\{[1 - x_0(s)]y_0(t) + x_0(s)y_1(t)\}, \end{aligned} \tag{4}$$

where

$$\omega = 1 - [x_1(s) - x_0(s)][y_1(t) - y_0(t)],$$
$$x_i(0) = y_i(0) = 0, x_i(1) = y_i(1) = 1, i = 0, 1.$$

This mapping transforms the rectangular grid on $K_0$ to the grid on $K$ consisting of the straight line segments connecting the images of the corresponding boundary points of $K_0$ which lie on the opposite sides of $K$.

The superposition of the mappings $u(x,y), v(x,y)$ and $x(s,t), y(s,t)$ is the resulting mapping $u(s,t)$, $v(s,t)$ of $K_0$ onto $\Omega$, which is defined in [1] as a unique solution of the variational problem given by the functional:

$$F_n = \frac{1}{2}\int_0^1\int_0^1 \frac{h_{22}(u_s^2 + v_s^2) - 2h_{12}(u_s u_t + v_s v_t) + h_{11}(u_t^2 + v_t^2)}{\sqrt{h_{11}h_{22} - h_{12}^2}} dsdt, \tag{5}$$

where $h_{ij} = h_{ij}(s, t, \vec{\mathbf{s}}, x_0, x_1, y_0, y_1)$ satisfy the condition $H = Q^* G Q$,

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix}, \quad G = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}. \tag{6}$$

The functional (5) is minimized with respect to three groups of arguments:

1) functions $u, v$;
2) metric parameters $\vec{s} = (s_1, s_2, s_3, s_4, k)$;
3) control functions $x_0, x_1, y_0, y_1$ or boundary point distributions.

The minimal value $F = S$, where $S$ is the area of $\Omega$, is achieved at the quasi-isometric mapping $u, v$.

At each side of $K_0$ a Dirichlet or free (natural) boundary condition for $u, v$ must be specified; in the case of a free boundary condition the corresponding control function must be given and the point distribution is to be found by the minimization process.

## 2.2 NUMERICAL METHOD

The implementation of the quasi-isometric mapping technique requires some special efforts described briefly in this item.

In the square $K_0$ we introduce a rectangular uniform $(s,t)$-grid with the step sizes $\Delta s$ and $\Delta t$ supposed to be constants. The finite-difference approximation of the functional (5) is standard. We will not use any discrete notation, we will write $(s,t)$, $x, y$, $x_s, x_t$, $y_s, y_t$, $u_s, u_t$, etc., keeping in mind corresponding grid objects; the discrete unknowns are the values of $x, y, u, v$ defined at the nodes of the $(s,t)$-grid. The image of the $(s,t)$ grid under the mapping $u(s,t), v(s,t)$ is the curvilinear $(u,v)$ grid in $\Omega$. For minimization (5) it is used an iterative process consisting of three stages in according to three groups of unknowns. Each stage is a minimization procedure with respect to one of these groups provided that the others are fixed on this stage. The functions $u, v$, represented the desired grid, are computed as a solution of the Euler variational equations for the integral (5) under the Dirichlet boundary conditions. The corresponding system of discrete elliptic equations is solved by multigrid method. The free boundary condition realize at the stage 3 as computation of boundary point distribution.

The other two minimization procedures are more complicated and require more detailed explanations.

**Computation of the metric parameters.**

Consider the matrices $\hat{H}$ and $\hat{G}$ with the elements $H_{ij}$ and $G_{ij}$ respectively, $1 \leq i, j \leq 2$:

$$H_{11} = u_s^2 + v_s^2, \quad H_{22} = u_t^2 + v_t^2, \quad H_{21} = H_{12} = u_s u_t + v_s v_t, \tag{7}$$

$$G_{11} = u_x^2 + v_x^2, \quad G_{22} = u_y^2 + v_y^2, \quad G_{21} = G_{12} = u_x u_y + v_x v_y. \tag{8}$$

The values $H_{ij}$ are computed at the mid-points of each $(s,t)$-cell, the values $G_{ij}$ are computed by means of the relation $\hat{G} = Q\hat{H}Q^*$, where $Q$ is defined in (6).

Using these notations we replace the functional (3) by

$$F = \int_0^1 \int_0^1 \frac{g_{11}G_{22} + g_{22}G_{11} - 2g_{12}G_{12} - 2\sqrt{g_{11}g_{22} - g_{12}^2}\sqrt{G_{11}G_{22} - G_{12}^2}}{2\sqrt{g_{11}g_{22}}\sqrt{G_{11}G_{22}}} dxdy. \tag{9}$$

here $dxdy = (x_s y_t - x_t y_s)dsdt$.

It is easy to show that the extremals of (3) and (9) coincide.

At each $(s,t)$-cell, consider new variables $(p,\varphi)$ and $(P,\Phi)$ defined by means of the relations:

$$\begin{aligned} g_{11} &= \rho e^{p}, \quad g_{22} = \rho e^{-p}, \quad g_{12} = \rho \sin\varphi, \\ G_{11} &= R e^{P}, \quad G_{22} = R e^{-P}, \quad G_{12} = R \sin\Phi. \end{aligned} \tag{10}$$

Note, the functional (9) does not depend on the values $\rho, R$. Using these new variables we transform the functional (9) to the form:

$$F = \frac{1}{2}\int_0^1\int_0^1 (2sh^2\frac{p-P}{2} + 2\sin^2\frac{\varphi-\Phi}{2})dxdy, \tag{11}$$

which has simple geometric interpretation as some distance between the desired metric $g_{ij}$ and the real metric $G_{ij}$, computed with respect to the functions $u, v$.

The functional (11) is non-negative and convex, if $|\varphi - \Phi| \le \pi/2$. To provide this inequality (even if the grid $(u,v)$ is bad) we realize a special regularization:

$$\sin\hat{\Phi} = \begin{cases} max\{\sin\Phi, -|\cos\varphi|\}, & \text{if } \sin\varphi \ge 0, \\ max\{\sin\Phi, |\cos\varphi|\}, & \text{if } \sin\varphi < 0, \end{cases} \tag{12}$$

and replace $\Phi$ by $\hat{\Phi}$ in all formulas.

A Newton-type procedure is used for minimization (11). This functional is approximated at the $(s,t)$ grid by the quadratic form with respect to the variation $\delta\vec{s}$ of the metric parameter $\vec{s}$. To provide the convenient and reliable computation we use instead of $\delta\vec{s}$ some new variables $\vec{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ connected linearly with $\delta\vec{s}$ and presented the variations of four angles and some analog of the conformal modulus. In this new notation we write

$$F \cong F_0 + (b, \vec{\xi}) + \frac{1}{2}(A\vec{\xi}, \vec{\xi}),$$

where respectively $b$ and $A$ is the vector and the matrix computed from the quadratic approximation of (11).

**Computation of the control functions.**

Replace the functional (5) by the equivalent one

$$F = \frac{1}{2}\int_0^1\int_0^1 \frac{h_{22}H_{11} - 2h_{12}H_{12} + h_{11}H_{22} - 2\sqrt{h_{11}h_{22} - h_{12}^2}H_0}{\sqrt{H_{11}H_{22}}\sqrt{h_{11}h_{22}}}dsdt$$

where $h_{ij}$ and $H_{ij}$ are defined by (5) and (7) respectively, $H_0 = u_s v_t - u_t v_s$.

Introduced new variables $p, \varphi, P, \Phi$ by relations:

$$h_{11} = \rho e^p, \qquad h_{22} = \rho e^{-p}, \qquad h_{12} = \rho \sin\varphi,$$

$$H_{11} = Re^P, \quad H_{22} = Re^{-P}, \quad H_{12} = R\sin\Phi,$$

we rewrite (13) similar to (9) in the form $F = 0.5\int_0^1\int_0^1 f ds dt$ with the integrand

$$f = 2sh^2\frac{p-P}{2} + 2\sin^2\frac{\varphi-\Phi}{2}.$$

Again we correct the field $(P, \Phi)$ by means of the regularization (12). Then, at each $(s,t)$-cell we approximate the integrand $f$ by the quadratic form with respect to the values of control functions and their derivatives at the mid-point of the cell. We split the computation into two steps according to the pairs of $x_0, x_1$ and $y_0, y_1$. For computing the functions $x_0, x_1$ consider the following transformations:

$$(p,\varphi) \xrightarrow{Q_1} (h_{11}, h_{22}, h_{12}) \xrightarrow{Q_2} (x, y, x_s, x_t, y_s, y_t) \xrightarrow{Q_3} (x_0, x_0', x_1, x_1').$$

Denote $\vec{z} = (z_1, z_2, z_3, z_4) \equiv (x_0, x_0', x_1, x_1')$. Twice differentiating the mappings $Q_1, Q_2, Q_3$ we find at each $(s,t)$-cell:

$$grad\ f \equiv (\frac{\partial f}{\partial z_i},\ i = \overline{1,4}), \quad C \equiv (\frac{\partial^2 f}{\partial z_i \partial z_k},\ i,k = \overline{1,4}),$$

and can write approximately

$$f \cong f_0 + (grad\ f, \delta\vec{z}) + \frac{1}{2}(C\delta\vec{z}, \delta\vec{z}).$$

Summing over $t$ at each fixed $s$ (mid-point of cell) we have the one-dimensional quadratic functional with respect to $\delta\vec{z}(s)$. The minimum condition is written for the functions $x_0, x_1$ defined at the grid nodes and represents the system of two three point discrete equations, which is solved by matrix sweep method; the boundary conditions are $x_i(0) = 0,\ x_i(1) = 1, i = 1,2$. The functions $y_0, y_1$ are computed by the same way.

If the position of the nodes on a boundary of the quadrangle $\Omega$ is unknown then the corresponding control function is to be given and not changed in the whole iterative process. The computation of the free boundary points is based on the algorithm of computation of the control functions. The approximation of the corresponding control function, obtained by means of above-mentioned algorithm, is used for computing new position of the boundary nodes by some interpolation formula.

# 3 MULTIBLOCK GRID GENERATION

There are not well-justified approaches to automatic block-structured quadrangular grid generation on a plane or a surface in space. To seek a solution of this problem we have been developing an approach based on the theory of the quasi-isometric mapping. Consider a physical domain $D$ which is to be subdivided into a finite number $N$ of quadrangular block $D_n, n = 1, 2, ..., N$. A block-structure of $D$ is defined by a topological description. Each block is formed by four boundaries, each of them may be either outer boundary or an interface of two blocks. Note the grid lines crossing any block interface are continuous. As a rule the initial positions of the interfaces apriori are unknown. These interfaces, as well as block-structured grid itself, are to be constructed by grid generation process using some topological and geometrical information.

Topological information consists of the description of block correspondence. Geometrical information includes the specification of the position of all outer boundaries, grid size of each blocks, some angles of the boundary intersections, some rules of boundary point distributions, position of interior block vertices being apriori fixed. This information is not usually sufficient to construct the whole block structure (included the interfaces) by a simple method.

The proposed method is based on construction both of the grid and the interfaces by means of discretization of the mapping of the union of the unit squares $K_0^{(n)}, n = \overline{1, N}$ onto $D$. Each square $K_0^{(n)}$ in the $(s, t)$ plane is mapped quasi-isometrically onto some subdomain $D^{(n)}$.

As a base quasi-isometric mapping it is used one described in Section 2, defined by the five-parameter family of metrics

$$g_{ij}^{(n)} = g_{ij}^{(n)}(x, y, \vec{s}^{(n)})$$

in the unit square $K^{(n)}$ in the $(x, y)$ plane.

The full algorithm is based on minimization of the total variational functional

$$F = F^{(1)} + F^{(2)} + ... + F^{(N)}$$

under given boundary conditions and some metric relations, here $F^{(n)}$ is the functional (5) written for the square $K_0^{(n)}$ in the $(s, t)$ plane. As in 2.2, the minimization process consists of three stages. The first one is the computation of the grid functions $\{(u, v)^{(n)}, n = \overline{1, N}\}$ under given the metric parameters and the control functions in each square $K_0^{(n)}$. The corresponding system of discrete equations in the whole block-domain is solved by multigrid method. It is developed a special technique to take into account irregular stencils of discretization arising in the block vertices.

The metric relations arise from requirement: each of block must be mapped on the same constant curvature surface. Shwartz type procedure is implemented to provide these relations.

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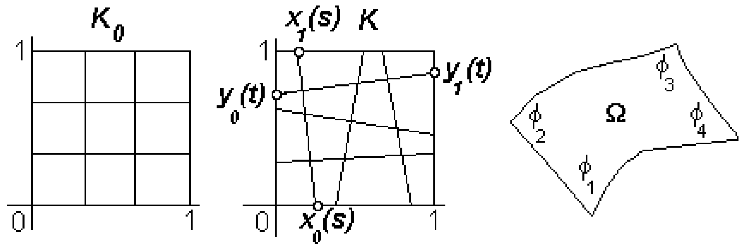


Fig. 1.

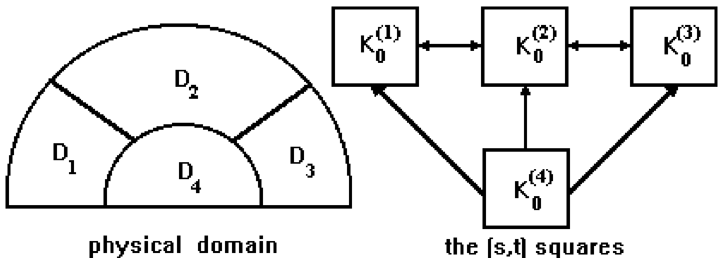


Fig. 2

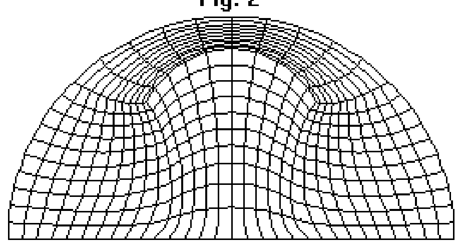

Fig. 3