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ON SYMMETRIC AND FUNCTIONAL RESTRICTIONS FOR CLASSIFICATION ALGORITHMS

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In this note we present results obtained in the framework of the algebraic approach ([1], [2]) to the recognition (classification) problem. In [3] and [4] the study of the completeness (solvability) of classification problems was reduced to the study of special (called *complete and admissible*) subcategories of the categories $\Psi^{q,l}$ (q and l are positive integers). The categories $\Psi^{q,l}$ are complete subcategories of the category of sets, and have as objects the spaces of $q \times l$ matrices over arbitrary sets and all finite Cartesian powers of such spaces.

Recognition problems can be posed as problems of designing algorithms that realize mappings from one given space (the space of possible initial information) into another (the space of possible final information), where the mappings realized must satisfy certain restrictions (each system of such restrictions determines a corresponding problem).

The systems of restrictions under consideration breaks up in a natural way into pairs of subsystems containing universal and local restrictions. The class of complete admissible categories that are subcategories of the categories $\Psi^{q,l}$ contains categories corresponding to all possible universal restrictions for classification problems. The conditions defining complete admissible categories are the most general restrictions that allow the completeness problem to be treated at all. At the same time the generality of the definition of complete admissible categories does not permit them to be investigated in sufficient detail. This moves us to pose the question of singling out in the class of complete admissible categories subclasses that are sufficiently narrow to admit thorough investigation and at the same time sufficiently broad to cover the whole of or at least most of the spectrum of the categories used in practice. The classes of symmetric and functional categories are such subclasses, and the present note is devoted to their study.

Let A and B be arbitrary sets, and Ψ a subcategory of the category $\Psi^{q,l}$ (it will be assumed that q and l are arbitrary fixed positive integers). The set $\mathrm{Hom}_\Psi(\mathfrak{C}(A), \mathfrak{C}(B))$ of morphisms, where $\mathfrak{C}(A)$ and $\mathfrak{C}(B)$ are the spaces of $q \times l$ matrices over A and B, will be denoted by $H_\Psi(A, B)$, and the set $\bigcup_{p=0}^{\infty} \mathrm{Hom}_\Psi(\mathfrak{C}^p(A), \mathfrak{C}(B))$ will be denoted by $\mathscr{H}_\Psi(A, B)$. For any $X \subseteq \mathfrak{C}(A)$ let

$$H_\Psi(A,B)(X) = \{u(x) | x \in X, u \in H_\Psi(A,B)\},$$

$$\mathscr{H}_\Psi(A,B)(X) = \bigcup_{p=0}^{\infty} \{u(x_1, \ldots, x_p) | (x_1, \ldots, x_p) \in X^p, \\ u \in \mathrm{Hom}_\Psi(\mathfrak{C}^p(A), \mathfrak{C}(B))\}.$$

DEFINITION 1. Let Ψ be a subcategory of $\Psi^{q,l}$. The category Ψ is said be *complete* if $\mathscr{H}_\Psi(A,B)(\mathfrak{C}(A)) = \mathfrak{C}(B)$ for any sets A and B with $|A| > 1$.

REMARK. Complete categories are not in general full subcategories of $\Psi^{q,l}$.

Let $u \colon A^p \to B$, where A and B are arbitrary sets and p is a positive integer. The *diagonalization* of u is defined to be the mapping $u_\triangle : A \to B$ such that $u_\triangle(x) = u(x, \ldots, x)$ for all $x \in A$.

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DEFINITION 2. A subcategory Ψ of the category $\Psi^{q,l}$ is said to be *admissible* if for any two morphisms u and v of Ψ, where $u\colon \mathfrak{C}^{p_1}(A) \to \mathfrak{C}^{p_2}(B)$ and $v\colon \mathfrak{C}^{p_3}(A) \to \mathfrak{C}^{p_4}(B)$, the product $u \times v$ and the diagonalization u_Δ are also morphisms of Ψ.

The concept of a base is central in the study of completeness of classification problems.

DEFINITION 3. Let A be an arbitrary nonsingleton set, and let $X \subseteq \mathfrak{C}(A)$. The set X of matrices is called a *base for the category* Ψ if $\mathscr{H}_\Psi(A,A)(X) = \mathfrak{C}(A)$.

The symbol σ_0 denotes the symmetric group of permutations of the set $S = \{(i,j) | i \in \{1,\dots,q\},\ j \in 1,\dots,l\}\}$.

Let A be an arbitrary set, a a matrix in $\mathfrak{C}(A)$, and s a permutation in σ_0. The definition of the action of the permutation s on the matrix a is given by the equality

$$s(a) = s(\|\alpha_{ij}\|_{q\times l}) = \|\alpha'_{ij}\|_{q\times l},$$

where $\alpha'_{ij} = \alpha_{s(i,j)}$ for all $(i,j) \in S$. This defines an action of s on $\mathfrak{C}(A)$.

We also define an action of s on $\mathfrak{C}^p(A)$, where p is an arbitrary positive integer:

$$s(a_1,\dots,a_p) = (s(a_1),\dots,s(a_p))$$

(here $(a_1,\dots,a_p)$ is an arbitrary set of matrices in $\mathfrak{C}^p(A)$).

Now let σ be a subgroup of σ_0. We associate with it a subcategory Σ of $\Psi^{q,l}$, setting $\operatorname{Ob}\Sigma = \operatorname{Ob}\Psi^{q,l}$ and defining (for arbitrary sets A and B and arbitrary positive integers p_1 and p_2) the morphism set $\operatorname{Hom}_\Sigma(\mathfrak{C}^{p_1}(A), \mathfrak{C}^{p_2}(B))$ to be the set of all mappings from $\mathfrak{C}^{p_1}(A)$ to $\mathfrak{C}^{p_2}(B)$ that commute with all the permutations in σ, i.e., the set of all mappings u such that for all $s \in \sigma$ and $(a_1,\dots,a_{p_1}) \in \mathfrak{C}^{p_1}(A)$

$$u(s(a_1,\dots,a_{p_1})) = s(u(a_1,\dots,a_{p_1})).$$

LEMMA 1. *For any subgroup σ of σ_0 the category Σ is complete and admissible.*

Symmetric categories formalize information about the homogeneity of the various objects and classes in classification problems. The functional categories described below are a formalization of information about simultaneous homogeneity and independence.

DEFINITION 4. A *functional signature* φ is defined to be a collection $(S_{(1,1)},\dots, S_{(q,l)}, \lambda)$ of linearly ordered subsets $S_{(i,j)}$ of the set S and a function $\lambda\colon S \to \{1,\dots,t\}$, where t is a positive integer no greater than ql. Further, for any (i_1,j_1) and (i_2,j_2) in S

$$(\lambda(i_1,j_1) = \lambda((i_2,j_2)) \Rightarrow (z(i_1,j_1) = z(i_2,j_2)),$$

where $z(i,j)$ is the cardinality of the set $S_{(i,j)}$.

The sets $S_{(i,j)}$ will be written as tuples $(s(i,j,1), s(i,j,2),\dots,s(i,j,z(i,j)))$, where the elements are written according to the order on $S_{(i,j)}$.

Suppose that $\varphi = (S_{(1,1)},\dots,S_{(q,l)},\lambda)$ is a functional signature, A and B are sets, and p_1 and p_2 are positive integers. We define the set $\Phi(\mathfrak{C}^{p_1}(A), \mathfrak{C}^{p_2}(B))$ as the set of all mappings from $\mathfrak{C}^{p_1}(A)$ to $\mathfrak{C}^{p_2}(B)$ that can be represented as follows with the help of the p_2t functions f^r_k (where $r \in \{1,\dots,p_2\}$ and $k \in \{1,\dots,t\}$):

$$u(a_1,\dots,a_{p_1}) = u(\|\alpha^1_{ij}\|_{q\times l},\dots,\|\alpha^{p_1}_{ij}\|_{q\times l}) = (\|\beta^1_{ij}\|_{q\times l},\dots,\|\beta^{p_2}_{ij}\|_{q\times l}),$$

where $\beta^r_{ij} = f^r_{\lambda(i,j)}(\alpha^1_{s(i,j,1)},\dots,\alpha^1_{s(i,j,z(i,j))},\dots,\alpha^{p_1}_{s(i,j,z(i,j))})$ for all $(a_1,\dots,a_{p_1}) \in \mathfrak{C}^{p_1}(A)$.

The sets $\Phi(\mathfrak{C}^{p_1}(A), \mathfrak{C}^{p_2}(B))$ of mappings cannot always be regarded as sets of morphisms of a specific subcategory of $\Psi^{q,l}$. This is possible only for the admissible functional signatures defined below.

DEFINITION 5. A functional signature $\varphi = (S_{(1,1)},\dots,S_{(q,l)},\lambda)$ is said to be *admissible* if $(i,j) \in S_{(i,j)}$ for all $(i,j) \in S$,

$$(\lambda(i_1,j_1) = \lambda(i_2,j_2))\&((i_1,j_1) = s(i_1,j_1,k)) \Rightarrow ((i_2,j_2) = s(i_2,j_2,k))$$

for all (i_1, j_1) and (i_2, j_2) in S and all k in $\{1, \ldots, z(i_1, j_1)\}$,

$$(\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow (\lambda(s(i_1, j_1, k)) = \lambda(s(i_2, j_2, k)))$$

for all (i_1, j_1) and (i_2, j_2) in S and all k in $\{1, \ldots, z(i_1, j_1)\}$,

$$((i_1, j_1) \in S_{(i_2, j_2)}) \Rightarrow (S_{(i_1, j_1)} \subseteq S_{(i_2, j_2)})$$

for all (i_1, j_1) and (i_2, j_2) in S, and

$$(\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow ((s(s(i_1, j_1, k), k_1) = s(i_1, j_1, k_2)) \equiv (s(s(i_2, j_2, k), k_1) = s(i_2, j_2, k_2)))$$

for all (i_1, j_1) and (i_2, j_2) in S, all k and k_2 in $\{1, \ldots, z(i_1, j_1)\}$, and all k_1 in $\{1, \ldots, z(s(i_1, j_1, k))\}$ (under the two preceding conditions).

LEMMA 2. *A functional signature φ determines a subcategory Φ of the category $\Psi^{q,l}$ if and only if φ is admissible.*

LEMMA 3. *For any admissible functional signature φ the corresponding category Φ is complete and admissible.*

THEOREM 1. *Let σ be a subgroup of the symmetric group σ_0, and let X be a subset of the space $\mathfrak{C}(A)$, where A is some arbitrary set. The set X is a base for the category Σ if and only if for any nonidentity permutation s in σ there is a matrix a in X such that $s(a) \neq a$.*

THEOREM 2. *Suppose that φ is an admissible functional signature and X is a subset of the space $\mathfrak{C}(A)$, where A is an arbitrary set. The set X is a base for the category Φ if and only if for any distinct (i_1, j_1) and (i_2, j_2) in S with $\lambda(i_1, j_1) = \lambda(i_2, j_2)$ there is a matrix $a = \|\alpha_{ij}\|_{q \times l}$ in X such that $\alpha_{s(i_1, j_1, k)} \neq \alpha_{s(i_2, j_2, k)}$ for some $k \in \{1, \ldots, z(i_1, j_1) = z(i_2, j_2)\}$.*

Each functional category is a subcategory of the corresponding symmetric categories. Further, morphisms of functional categories are easy to give explicitly in an obvious way (to determine them it suffices to describe tuples of functions), while morphisms of symmetric categories are determined implicitly as mappings satisfying a certain system of restrictions. In this connection the question arises of the possibility of solving classification problems with universal restrictions expressed by symmetric categories by using only morphisms of functional subcategories of such categories. This question will be solved below.

DEFINITION 6. Let Ψ_1 and Ψ_2 be subcategories of the category $\Psi^{q,l}$, with Ψ_1 a subcategory of Ψ_2. The category Ψ_1 is said to be a Γ-*complete subcategory of* Ψ_2 if each base for Ψ_2 is also a base for Ψ_1.

DEFINITION 7. For an admissible functional signature $\varphi = (S_{(1,1)}, \ldots, S_{(q,l)}, \lambda)$, the *group* σ_φ is defined to be the subgroup of σ_0 consisting of all the permutations s_φ satisfying the conditions

$$\lambda(s_\varphi(i, j)) = \lambda(i, j)$$

for all (k, j) in S, and

$$s_\varphi(s(i, j, k)) = s(s_\varphi(i, j), k)$$

for all (i, j) in S and all k in $\{1, \ldots, z(i, j)\}$.

THEOREM 3. *Let φ be an admissible functional signature, and σ a subgroup of σ_0. The category Φ is a subcategory of the category Σ if and only if the group σ_φ is a subgroup of σ.*

THEOREM 4. *Let $\varphi = (S_{(1,1)}, \ldots, S_{(q,l)}, \lambda)$ be an admissible functional signature. The category Φ is a Γ-complete subcategory of the category Σ_φ corresponding to the* *subgroup σ_φ if and only if the signature φ satisfies the condition*

$$(S_{(i_1,j_1)} \subseteq S_{(i_2,j_2)}) \& (S_{(i_1,j_1)} \neq S_{(i_2,j_2)}) \Rightarrow (|\lambda^{-1}(\lambda(i_1,j_1))| = 1)$$

for all (i_1, j_1) and (i_2, j_2) in S, where $\lambda^{-1}(\lambda(i_1, j_1))$ is the kernel equivalence class for λ that contains (i_1, j_1), and $|\lambda^{-1}(\lambda(i_1, j_1))|$ is the cardinality of this class.

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