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Completeness Criteria for Models of Algorithms and Decision Rule Classes in Classification Problems with Set-Theoretic Constraints

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In the framework of the algebraic approach to the synthesis of correct algorithms for pattern recognition, classification, and prediction [1, 2], we consider a class of problems characterized by explicit set-theoretic constraints imposed on the admissible output space of an algorithm.

Following [3], the classification problem is described as the problem of designing a data-transformation algorithm. Consider a set $\mathcal{S} = \{S\}$, whose elements are called objects. The descriptions $D(S)$ of the objects form the initial-information space $\mathfrak{I}_i = \{D(S) | S \in \mathcal{S}\}$, whose elements are denoted by I_i, so that $\mathfrak{I}_i = \{I_i\}$.

Consider the problem of designing algorithms A that implement mappings from $\mathfrak{I}_i$ to the final-information space $\mathfrak{I}_f = \{I_f\}$. In what follows, we do not distinguish algorithms and the mappings they implement. A solution is synthesized within the framework of a model $\mathfrak{M}$ of algorithms, where $\mathfrak{M} \subseteq \{A | A: \mathfrak{I}_i \to \mathfrak{I}_f\}$. An individual problem is defined by structural information I_s that singles out from $\mathfrak{M}$ a subset of admissible mappings, designated as $\mathfrak{M}[I_s]$. Any algorithm A implementing an arbitrary admissible mapping is called correct for the problem defined by I_s and is its solution.

Constructions based on the algebraic approach to the synthesis of correct algorithms use an estimate space $\mathfrak{I}_e = \{I_e\}$ that is intermediate between $\mathfrak{I}_i$ and $\mathfrak{I}_f$. Correct algorithms are synthesized on the basis of heuristic information models (i.e., parametric classes of mappings from $\mathfrak{I}_i$ to $\mathfrak{I}_f$), algorithmic operators representing a special superposition (mappings from $\mathfrak{I}_i$ to $\mathfrak{I}_e$), and decision rules (mappings from $\mathfrak{I}_e^p$ to $\mathfrak{I}_f$, p is the arity of a decision rule).

Recall that, for arbitrary sets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, and $\mathcal{V}'$ and arbitrary mappings u from $\mathcal{U}$ to $\mathcal{V}$ and u' from $\mathcal{U}'$ and $\mathcal{V}'$, the product $u \times u'$ is a mapping v of $\mathcal{U} \times \mathcal{U}'$ to $\mathcal{V} \times \mathcal{V}'$ such that $v(U, U') = (u(U), u(U'))$ for any pair (U, U') from $\mathcal{U} \times \mathcal{U}'$ [4]. For an arbitrary mapping u from $\mathcal{U}^p$ to $\mathcal{V}$ with $p \geq 1$, a diagonalization u_Δ is a mapping from $\mathcal{U}$ to $\mathcal{V}$ such that $u_\Delta(U) = u(U, U, \ldots, U)$ for any U from $\mathcal{U}$.

The models $\mathfrak{M}$ are defined by models of algorithmic operators $\mathfrak{M}^0$, where $\mathfrak{M}^0 \subseteq \mathfrak{M}_* \stackrel{\text{def}}{=} \{B | B: \mathfrak{I}_i \to \mathfrak{I}_e\}$, and by decision rules $\mathfrak{M}^1$, where $\mathfrak{M}^1 \subseteq \bigcup_{p=0}^{\infty} \{C | C: \mathfrak{I}_e^p \to \mathfrak{I}_f\}$, as follows:

$$\mathfrak{M} = \mathfrak{M}^1 \circ \mathfrak{M}^0 = \{C \circ (B_1 \times B_2 \times \ldots \times B_p)_\Delta |,$$
$$C \in \mathfrak{M}^1, B_1, B_2, \ldots, B_p \in \mathfrak{M}^0\}.$$

Along with the set of mappings $\mathfrak{M}_*$ defined above, a set $\mathfrak{F}$ of correcting operations is also used for designing correct algorithms. The correcting operations F considered here are induced by operations F over $\mathfrak{I}_e$:

$$F(B_1, B_2, \ldots, B_p)(I_i) \stackrel{\text{def}}{=} F(B_1(I_i), B_2(I_i), \ldots, B_p(I_i)),$$

where I_i ranges over $\mathfrak{I}_i$, the algorithmic operators B_1, B_2, ..., B_p are arbitrary mappings from $\mathfrak{I}_i$ to $\mathfrak{I}_e$, and F is an operation over $\mathfrak{I}_e$.

The construction scheme for an algorithm model $\mathfrak{M}$ is shown in the following commutative diagram [3]:

$$\begin{array}{ccc} \mathfrak{I}_i & \xrightarrow{\mathfrak{M}} & \mathfrak{I}_f \\ \mathfrak{M}^0 \downarrow & & \uparrow \mathfrak{M}^1 \\ \mathfrak{I}_e^p & \xrightarrow[\mathfrak{F}]{} & \mathfrak{I}_e \end{array}.$$

For the problems with set-theoretic constraints considered here, algorithm models $\mathfrak{M}$ are constructed on the basis of parametric classes of models of algorithmic operators and correcting operations. It is assumed that $\mathfrak{M}^0 = \{\mathfrak{M}^0_{\lambda, \omega} | \lambda \in L, \omega \in W(\lambda)\}$ and $\mathfrak{F} = \{\mathfrak{F}^\lambda | \lambda \in L\}$,

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where $W(\lambda)$ and L are sets of structural indices. A model $\mathfrak{M}$ is constructed in the form

$$\mathfrak{M} = \bigcup_{\lambda \in L} \bigcup_{\omega \in W(\lambda)} \mathfrak{M}^1 \circ \mathfrak{F}^{\lambda}(\mathfrak{M}^0_{\lambda,\omega}),$$

where

$$\mathfrak{M}^1 \circ \mathfrak{F}^{\lambda}(\mathfrak{M}^0_{\lambda,\omega}) = \{C \circ F_1((B^1_1, B^1_2, \ldots, B^1_{r(1)}) \ldots$$

$$\ldots \times F_p(B^p_1, B^p_2, \ldots, B^p_{r(p)}))_{\Delta} \mid C \in \mathfrak{M}^1,$$

$$(F_1, F_2, \ldots, F_p) \in (\mathfrak{F}^{\lambda})^p, B^1_1, B^1_2, \ldots, B^1_{r(1)} \in \mathfrak{M}^0_{\lambda,\omega(1)}, \ldots$$

$$\ldots, B^p_{r(p)}, \ldots, B^p_{r(p)} \in \mathfrak{M}^0_{\lambda,\omega(p)}\}$$

for all $\lambda \in L$ and $\omega \in W(\lambda)$.

To formalize the concept of set-theoretic constraints, we introduce a set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ of predicates $\pi_i: \mathfrak{I}_i \times \mathfrak{I}_f \to \{0, 1\}$.

Let I_i be an arbitrary element of $\mathfrak{I}_i$. Let $\Pi(I_i) = \left\{ I_f \mid I_f \in \mathfrak{I}_f, \underset{1,\ldots,k}{\forall} j: \pi_j(I_i, I_f) = 1 \right\}$ be the set of all admissible values of correct algorithms for initial information I_i.

A set Π is called covering if $\Pi(I_i) \neq \emptyset$ for any I_1 in $\mathfrak{I}_i$, i.e., if for any element, there exists at least one admissible value.

In what follows, we consider an arbitrary fixed covering set Π of predicates.

Denote the set of positive integers by N and set $N_0 = N \cup \{0\}$.

Definition 1. The set

$$\text{Prec} = \{((I^1_i, I^2_i, \ldots, I^q_i), (I^1_f, I^2_f, \ldots, I^q_f)) \mid$$

$$q \in N, (I^1_i, I^2_i, \ldots, I^q_i) \in \mathfrak{I}^q, I^j_i \neq I^k_i \quad \text{for} \quad j \neq k,$$

$$(I^1_f, I^2_f, \ldots, I^q_f) \in \mathfrak{I}^q_f, I^j_f \in \Pi(I^j_i)$$

$$\text{for} \quad j = 1, 2, \ldots, q\}$$

is the set of collections of admissible precedents.

For an arbitrary set $\mathfrak{I}$ and $q \in N$, the symbol $(\mathfrak{I}^q)^*$ stands for the set $(I^1, I^2, \ldots, I^q) \mid (I^1, I^2, \ldots, I^q) \in \mathfrak{I}^q, I^k \neq I^j$ for $k \neq j\}$.

Note that $\text{Prec} = \bigcup_{q \in N} \bigcup_{(I^1_i, \ldots, I^q_i) \in (\mathfrak{I}^q_i)^*} \{(I^1_i, I^2_i, \ldots, I^q_i),$ $\Pi(I^1_i) \times \Pi(I^2_i) \times \ldots \times \Pi(I^q_i)$.

Definition 2. A model $\mathfrak{M}$ is called Π-complete if

$$\underset{\mathfrak{I}_i}{\forall} I_i: \mathfrak{M}(I_i) = \{A(I_i) \mid A \in \mathfrak{M}\} \subseteq \Pi(I_i); \quad (1)$$

$$\underset{\text{Prec}}{\forall}((I^1_i, I^2_i, \ldots, I^q_i), (I^1_f, I^2_f, \ldots, I^q_f)),$$

$$\underset{\mathfrak{M}}{\exists} A: \underset{\{1,\ldots,q\}}{\forall} j: A(I^j_i) = I^j_f. \quad (2)$$

Note that conditions (1) and (2) are independent. Moreover, under condition (2), condition (1) is equivalent to

$$\underset{\mathfrak{I}_i}{\forall} I_i: \mathfrak{M}(I_i) = \{A(I_i) \mid A \in \mathfrak{M}\} = \Pi(I_i). \quad (1')$$

The analysis of the completeness problem in the framework of the algebraic approach is aimed at finding the conditions on $\mathfrak{M}^1$, $\mathfrak{F}$, and $\mathfrak{M}^0$ under which the model $\mathfrak{M} = \bigcup_{\lambda \in L} \bigcup_{\omega \in W(\lambda)} \mathfrak{M}^1 \circ \mathfrak{F}^{\lambda}(\mathfrak{M}^0_{\lambda,\omega})$ is complete.

It can easily be seen that the completeness problem for $\mathfrak{M}$ can be analyzed under the assumption that q is equal to 1. Indeed, to this end, it suffices to proceed from $\mathfrak{I}_i$ to $\bigcup_{q=1}^{\infty} \mathfrak{I}^q_i$, from $\mathfrak{I}_f$ to $\bigcup_{q=1}^{\infty} \mathfrak{I}^q_f$, from $\mathfrak{I}_e$ to $\bigcup_{q=1}^{\infty} \mathfrak{I}^q_e$, and from the original mappings (say, $A \in \mathfrak{M}$, $A: \mathfrak{I}_i \to \mathfrak{I}_f$) to $A^*: \bigcup_{q=1}^{\infty} \mathfrak{I}^q_i \to \bigcup_{q=1}^{\infty} \mathfrak{I}^q_f$, where $A^*(I^1_i, I^2_i, \ldots, I^q_i) \overset{\text{def}}{=} (A(I^1_i), A(I^2_i), \ldots, A(I^q_i))$.

Definition 3. A family of decision rules $\mathfrak{M}^1$ is called Π-complete if there exists a model of algorithmic operators $\mathfrak{M}^0$ and a family of correcting operations $\mathfrak{F}$ such that $\mathfrak{M} = \bigcup_{\lambda \in L} \bigcup_{\omega \in W(\lambda)} \mathfrak{M}^1 \circ \mathfrak{F}^{\lambda}(\mathfrak{M}^0_{\lambda,\omega})$ is a Π-complete model.

Consider a nonempty decision rule family $\mathfrak{M}^1 = \bigcup_{p=0}^{\infty} \mathfrak{M}^1_p$, where $\mathfrak{M}^1_p \subseteq \{C \mid C: \mathfrak{I}^p_e \to \mathfrak{I}_f\}$ for any p in N_0. For any $X \subseteq \mathfrak{I}_e$, it turns out that

$$\mathfrak{M}^1(X) = \bigcup_{p=0}^{\infty} \mathfrak{M}^1_p(X^p) \bigcup_{p=0}^{\infty} \bigcup_{C \in \mathfrak{M}^1_p} \bigcup_{\bar{x} \in X^p} C(\bar{x}).$$

Definition 4. Let $p \in N_0$. For an arbitrary I_i in $\mathfrak{I}_i$, the set $\alpha_p(\mathfrak{M}^1, I_i)$ is the intersection, in the pth Cartesian power of $\mathfrak{I}_e$, of all complete preimages of the set $\Pi(I_i)$ with respect to decision rules of arity p:

$$\alpha_p(\mathfrak{M}^1, I_i) = \bigcap_{C \in \mathfrak{M}^1_p} C^{-1}(\Pi(I_i))$$

$$= \left\{ \bar{I}_e \mid \bar{I}_e \in \mathfrak{I}^p_e, \quad \forall_{\mathfrak{M}^1_p} C\colon C(\bar{I}_e) \in \Pi(I_i) \right\}. \quad (3)$$

Definition 5. Let $p \in N_0$. For a family $\mathfrak{M}^1$ and an element I_i of $\mathfrak{I}_i$, a subset $X(I_i)$ of $\mathfrak{I}_e$ is called an admissible p-projection if

$$(X(I_i))^p \subseteq \alpha_p(\mathfrak{M}^1, I_i), \quad (4)$$

$$\neg \exists Z \subseteq \mathfrak{I}_e\colon (X(I_i) \subset Z) \wedge (Z^p \subseteq \alpha_p(\mathfrak{M}^1, I_i)). \quad (5)$$

The set of all admissible p-projections for $\mathfrak{M}^1$ and I_i is denoted by $\xi_p(\mathfrak{M}^1, I_i)$.

For an arbitrary I_i in $\mathfrak{I}_i$, we introduce the set $\Phi(\mathfrak{M}^1, I_i)$ of choice functions of admissible projections:

$$\Phi(\mathfrak{M}^1, I_i) = \{\varphi \mid \varphi\colon N_0 \to \mathrm{B}(\mathfrak{I}_e),$$

$$\forall_{N_0} p\colon ((\mathfrak{M}^1_p = \emptyset) \Rightarrow (\varphi(p) = \mathfrak{I}_e)) \wedge ((\mathfrak{M}^1_p \neq \emptyset)$$

$$\Rightarrow (\varphi(p) \in \xi_p(\mathfrak{M}^1, I_i)))\},$$

where $\mathrm{B}(\mathfrak{I}_e)$ is the set of all subsets of $\mathfrak{I}_e$.

For each choice function of admissible projections φ in $\Phi(\mathfrak{M}^1, I_i)$, we set $X(I_i, \varphi) = \bigcap_{p=0}^{\infty} \varphi(p)$. Note that

$$\mathfrak{M}^1(X(I_i, \varphi)) = \bigcup_{r=0}^{\infty} \bigcup_{C \in \mathfrak{M}^1_r} C\left(\left(\bigcap_{p=0}^{\infty} \varphi(p)\right)^r\right).$$

Let $\Phi'(\mathfrak{M}^1, I_i) = \{\varphi \mid \varphi \in \Phi(\mathfrak{M}^1, I_i), X(I_i, \varphi) \neq \emptyset\}$.

Theorem 1. *For all I_i in $\mathfrak{I}_i$,*

$$\bigcup_{\varphi \in \Phi'(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)) \subseteq \Pi(I_i). \quad (6)$$

Theorem 2 (Π-completeness criterion for decision rule classes). *A decision rule family $\mathfrak{M}^1$ is Π-complete if and only if*

$$\bigcup_{\varphi \in \Phi'(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)) = \Pi(I_i) \quad (7)$$

for any I_i in $\mathfrak{I}_i$.

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