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Algebraic Approach to the Problem of Synthesis of Trainable Algorithms for Trend Revealing

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As a formalization of the problem of synthesis trainable algorithms trends revealing, which is necessary for applying the algebraic approach of [1–4], we suggest the following construction.

Consider a set of finite plane configurations of the form $\bar{S}^d = (S^1, S^2, \ldots, S^d) = ((t^1, v^1), (t^2, v^2), \ldots, (t^d, v^d))$, where $t \in R$, $v \in R$, $t^1 \le t^2 \le \ldots \le t^d$, and, for $i = 1, 2, \ldots, d-1$, $d \ge 1$ $v^i < v^{i+1}$ whenever $t^i = t^{i+1}$. Endowing a finite plane configuration with a subscript α, we always assume that $\bar{S}_\alpha^{d(\alpha)} = (S_\alpha^1, S_\alpha^2, \ldots, S_\alpha^{d(\alpha)})$.

We denote the set of all d-point plane configurations by $C^d = \{\bar{S}^d\}$. We also define the set of all configurations as $C = \bigcup_{d=1}^{\infty} C^d$.

Configurations $\bar{S}_1^d$ and $\bar{S}_2^d$ are called shift-equivalent if there exists a vector $p^* = (t^*, v^*) \in R^2$ such that $S_1^i = S_2^i + p^* = (t_2^i + t^*, v_2^i + v^*)$ for all $i = 1, 2, \ldots, d$, where $S_1^i \in \bar{S}_1^d$ and $S_2^i \in \bar{S}_2^d$. In this case, we write $\bar{S}_1^d \cong \bar{S}_2^d$.

A marking vocabulary, or a set of marks, is, by definition, a finite set $M = \{\mu_1, \mu_2, \ldots, \mu_r\}$, where $r \ge 1$. For instance, a marking vocabulary may have the form $M = \{\max, \min, \text{non}\}$.

The set $M_\Delta = M \cup \{\Delta\}$, where $\Delta \notin M$ is a special mark interpreted as "not marked," is called the extended set of marks, or the extended marking vocabulary.

For a given set of marks M and the corresponding extended set of marks M_Δ, an arbitrary sequence $\bar{\mu}^d = (\mu^1, \mu^2, \ldots, \mu^d)$ of length $d \ge 1$ is called a marking of length d (if $\mu^i \in M$) or a partial marking of length d (if $\mu^i \in M_\Delta$).

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Note that it is the notion of marked configurations that allows us to state the training problem as the problem of synthesizing a correct algorithm for marking configurations.

We denote the set of all different marks of length d by M^d and the set of all different partial marks of length d by M_Δ^d. We also introduce the sets $\mathfrak{M} = \bigcup_{d=1}^{\infty} M^d$ and $\mathfrak{M}_\Delta = \bigcup_{d=1}^{\infty} M_\Delta^d$ of all different markings.

A marked configuration is a pair $(\bar{S}^d, \bar{\mu}^d)$, where $\bar{S}^d \in C^d$ and $\bar{\mu}^d \in M^d$. The marking $\bar{\mu}^d$ is then called a marking, or complete marking, of the configuration $\bar{S}^d$. For $\bar{\mu}^d \in M_\Delta^d$, the pair $(\bar{S}^d, \bar{\mu}^d)$ is called a partially marked configuration, and the marking $\bar{\mu}^d$ is then called a partial marking of the configuration $\bar{S}^d$.

A marking $\bar{\mu}^d = (\mu^1, \mu^2, \ldots, \mu^d)$ (complete or partial) of a configuration $\bar{S}^d$ is said to be an extension of a marking $\bar{\mu}_0^d = (\mu_0^1, \mu_0^2, \ldots, \mu_0^d)$ if, for any $i = 1, 2, \ldots, d$, the condition $(\mu_0^i = \mu^i) \vee (\mu_0^i = \Delta)$ holds.

Let us introduce a definition of the trend revealing algorithm as an algorithm for marking finite plane configurations.

Definition 1. A marking algorithm is an arbitrary algorithm A implementing a mapping $A: C \to \mathfrak{M}$ such that, for any $d \ge 1$, we have $A(\bar{S}^d) = \bar{\mu}^d$, where $\bar{S}^d \in C^d$ and $\bar{\mu}^d \in M^d$.

An essential special feature of this problem in comparison with the general problem of recognition and classifications, for which the algebraic approach was initially developed, is the presence of additional conditions (rules) relating geometric characteristics of configurations to possible "reasonable" markings [5].

Marking axioms, or rules, are defined as a set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ effectively computable predicates

$$\pi_i: \bigcup_{d=1}^{\infty}(C^d \times M^d) \to \{0, 1\}.$$

We use the same symbol Π to denote the conjunction of the predicates π_i:

$$\Pi = \bigwedge_{i=1,\ldots,k} \pi_i, \quad \Pi: \bigcup_{d=1}^{\infty}(C^d \times M^d) \to \{0, 1\}.$$

Suppose that a system of marking rules $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ is fixed. A marking $\bar{\mu}^d$ is called suitable for $\bar{S}^d$ if $\Pi(\bar{S}^d, \bar{\mu}^d) = 1$. A partial marking $\bar{\mu}_0^d \in M_\Delta^d$ is called suitable for $\bar{S}^d$ if and only if there exists a complete suitable marking $\bar{\mu}^d$ being an extension of $\bar{\mu}_0^d$.

A system of marking rules $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ is said to be shift-stable if and only if

$$\forall_{C} \bar{S}_1^d, \bar{S}_2^d: (\bar{S}_1^d \cong \bar{S}_2^d)$$

$$\Rightarrow \forall_{A} \bar{\mu}^d: (\Pi(\bar{S}_1^d, \bar{\mu}^d) = 1) \Leftrightarrow (\Pi(\bar{S}_2^d, \bar{\mu}^d) = 1).$$

A marking algorithm A is called shift-stable if and only if it identically marks any two shift-equivalent configurations, i.e.,

$$\forall_{C} \bar{S}_1^d, \bar{S}_1^d: (\bar{S}_1^d \cong \bar{S}_2^d) \Rightarrow A(\bar{S}_1^d) = A(\bar{S}_2^d).$$

Suppose that a system of marking rules $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ is fixed. Then the algorithms that yield only suitable markings for all $\bar{S}^d$ from C are called suitable algorithms.

In what follows, we assume that a marking vocabulary M and a shift-stable system of marking rules $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ are given; we consider only shift-stable suitable algorithms.

An arbitrary finite set of pairs of the form

$$H = \{\bar{S}_i^{d(i)}, \bar{\mu}_i^{d(i)} \mid \bar{S}^{d(i)} \in C^{d(i)};$$
$$\bar{\mu}_i^{d(i)} \in M_\Delta^{d(i)}; i = 1, 2, \ldots, q\}$$

is called a set of precedents.

Consider a set formed by shift-equivalent configurations $\bar{L} = (\bar{S}_1^d, \bar{S}_2^d, \ldots, \bar{S}_l^d)$ to which partial markings $\bar{L}^\mu = (\bar{\mu}_1^d, \bar{\mu}_2^d, \ldots, \bar{\mu}_l^d)$ are assigned.

Definition 2. A set $\bar{L}$ of partially marked shift-equivalent configurations of length d is called contradictory if and only if, for any complete marking $\bar{\mu}_*^d = (\mu_*^1, \mu_*^2, \ldots, \mu_*^d) \in M^d$ from marking suitability for all configurations from the set, it follows that there are two equivalent configurations where two marked points with the same number are marked differently; i.e.,

$$\left(\forall_{M^d} \bar{\mu}_*^d \ \exists_{\{1,2,\ldots,l\}} \alpha: \Pi(\bar{S}_\alpha^d, \bar{\mu}_*^d) = 1\right)$$

$$\Rightarrow \exists_{\{1,2,\ldots,d\}} i \ \exists_{\{1,2,\ldots,l\}} \beta: (\mu_\beta^i \neq \mu_*^i) \wedge (\mu_\beta^i \neq \Delta).$$

A set H containing no contradictory subsets of shift-equivalent configurations is called consistent.

This definition applies also to the case of partial markings of one configuration $\bar{S}_1^d = \bar{S}_2^d = \ldots = \bar{S}_l^d$, i.e., to the case where the subset contains several different partial markings of the same configuration.

In what follows, we assume that a set of precedents

$$H = \{(\bar{S}_i^{d(i)}, \bar{\mu}_i^{d(i)}) \mid \bar{S}^{d(i)} \in C^{d(i)}; \}.$$
$$\bar{\mu}_i^{d(i)} \in M_\Delta^{d(i)}; i = 1, 2, \ldots, q$$

is fixed.

The problem Z of trend revealing consists in synthesizing a suitable algorithm A such that, for all $i = 1, 2, \ldots, q$, the complete marking $A(\bar{S}_i^{d(i)})$ is an extension of the marking $\bar{\mu}_i^{d(i)}$; in other words, the algorithm should be such that

$$\forall_{\{1,2,\ldots,d(i)\}} j: (\mu_i^j \in \bar{\mu}_i^{d(i)}) \Rightarrow ((\mu_i^j \neq \Delta) \Rightarrow (\mu_i^j = \gamma_i^j)), \quad (1)$$

where $\bar{\gamma}_i^{d(i)} = A(\bar{S}_i^{d(i)})$.

Note that, if an algorithm A is suitable, then $\Pi(\bar{S}_i^{d(i)}, \bar{\gamma}_i^{d(i)}) = 1$ for all $\bar{\gamma}_i^{d(i)} = A(\bar{S}_i^{d(i)})$.

An algorithm A satisfying condition (1) is called correct for the problem Z.

Definition 3. The trend revealing problem Z is called solvable if and only if there exists a suitable correct algorithm A for this problem.

Theorem 1 (a solvability criterion). *The problem Z is solvable if and only if the set of precedents H is consistent.*

Definition 4. The problem Z is called regular if and only if Z is solvable for any suitable partial markings $\bar{\mu}_i^{d(i)} \in M_\Delta^{d(i)}$ of all configurations $\bar{S}_i^{d(i)}$ from H.

Theorem 2 (a regularity criterion). *The solvable problem Z is regular if and only if, for any subset* $\tilde{H} = \{(\bar{S}_p^{d(p)}, \bar{\mu}_p^{d(p)}) \mid p \in \{l_1, l_2, \ldots, l_u\} \subseteq \{1, 2, \ldots, q\}, u > 1\}$ *of H such that* $\bar{S}_i^{d(i)} \cong \bar{S}_j^{d(j)}$ *for any* $i, j \in \{l_1, l_2, \ldots, l_u\}$, *the marking rules imply the existence and uniqueness of a suitable marking of the configurations* $\bar{S}_p^{d(p)}$, *where* $p \in \{l_1, l_2, \ldots, l_u\}$.

Yet another important special feature of the problem under consideration is the necessity to formalize and apply the locality properties [6].

A subconfiguration $P(\bar{S}^d)$ of a configuration $\bar{S}^d$ is, by definition, an arbitrary subset of points from $\bar{S}^d$: $P(\bar{S}^d) \subseteq \bar{S}^d$. A subconfiguration $P_0(\bar{S}^d)$ of a configuration $\bar{S}^d$ is said to be connected if $P_0(\bar{S}^d) = (S^\alpha, S^{\alpha+1}, \ldots, S^{\beta-1}, S^\beta)$, where $1 \le \alpha \le \beta \le d$.

A neighborhood of a point $S^\iota \in \bar{S}^d$ is a pair formed by the point S^ι itself and some connected subconfiguration $P_0(\bar{S}^d)$ of the configuration $\bar{S}^d$ including this point:

$$O(S^\iota, \bar{S}^d) = (S^\iota, P_0(\bar{S}^d)) = (S^\iota, (S^\alpha, S^{\alpha+1}, \ldots, S^\beta)),$$
$$\text{where } 1 \le \alpha \le i \le \beta \le d.$$

The point S^ι is called the support point of the neighborhood $O(S^\iota, \bar{S}^d)$.

We say that a system of neighborhoods $O(\bar{S}^d)$ is given on a configuration $\bar{S}^d$, if each point $S^\iota \in \bar{S}^d$ is assigned some neighborhood $O(S^\iota, \bar{S}^d)$. A system of neighborhoods Ω is given on C a system of neighborhoods is given for each configuration from C.

A system of neighborhoods $O(\bar{S}^d)$ of a configuration $\bar{S}^d$ is called trivial if the neighborhood of each point coincides with the entire configuration, i.e.,

$$\underset{\bar{S}^d}{\forall} S^\iota: O(S^\iota, \bar{S}^d) = \bar{S}^d.$$

A system of neighborhoods Ω given on C is called trivial if and only if the system of neighborhoods of each configuration from C is trivial, i.e.,

$$\underset{C}{\forall} \bar{S}^d \underset{\bar{S}^d}{\forall} S^\iota: O(S^\iota, \bar{S}^d) = \bar{S}^d.$$

Suppose that a nontrivial system of neighborhoods Ω on C is given. An axiom $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ is called Ω-local if and only if

$$\underset{C}{\forall} \bar{S}_1^{d(1)}, \bar{S}_2^{d(2)} \underset{\bar{S}_1^{d(1)}}{\forall} S_1^\alpha \underset{\bar{S}_2^{d(2)}}{\forall} S_2^\beta:$$

$$(O(S_1^\alpha, \bar{S}_1^{d(1)}) \cong O(S_2^\beta, \bar{S}_2^{d(2)}))$$

$$\Rightarrow (\underset{M^b}{\forall} \bar{\mu}^b: \pi_\iota(S_1^\alpha, \bar{\mu}^b) = \pi_\iota(S_2^\beta, \bar{\mu}^b)),$$

where $b = \left|O(S_1^\alpha, \bar{S}_1^{d(1)})\right| = \left|O(S_2^\beta, \bar{S}_2^{d(2)})\right|$.

A system of axioms $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ is called Ω-local if and only if each axiom from Π is Ω-local.

Suppose that a system of neighborhoods Ω and a shift-stable Ω-local system of axioms $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ are given on C.

Definition 5. A marking algorithm A is called Ω-local if and only if, for any configurations $\bar{S}^{d(\alpha)}$, $\bar{S}^{d(\beta)} \in C$ and any $\alpha = 1, 2, \ldots, d(\alpha)$ and $\beta = 1, 2, \ldots, d(\beta)$,

$$(O(S^\alpha, \bar{S}^{d(\alpha)}) \cong O(S^\beta, \bar{S}^{d(\beta)})) \Rightarrow \mu^\alpha = \mu^\beta,$$

where $\mu^\alpha \in \bar{\mu}^{d(\alpha)} = A(\bar{S}^{d(\alpha)})$ and $\mu^\beta \in \bar{\mu}^{d(\beta)} = A(\bar{S}^{d(\beta)})$.

Note that the neighborhoods differ from the configurations in their structure. In essence, a neighborhood is a punctured configuration, i.e., a configuration in which a support point is fixed. For this reason, neighborhoods need their own definition of shift-equivalence: neighborhoods $O(S_1^\alpha, \bar{S}_1^d) = (S_1^\alpha, (S_1^{\alpha-k}, S_1^{\alpha-k+1}, \ldots, S_1^\alpha, \ldots, S_1^{\alpha+l}))$ and $O(S_2^\beta, \bar{S}_2^d) = (S_2^\beta, (S_2^{\beta-k}, \ldots, S_2^\beta, \ldots, S_2^{\beta+l}))$, where $k \ge 0$ and $l \ge 0$, are called shift-equivalent if there exists a vector $p^* = (t^*, v^*)$ such that $S_1^{\alpha+m} = S_2^{\beta+m} + p^*$ for all $m \in \{-k, -k+1, \ldots, l\}$.

Note that, at $m = 0$, the shift-equivalence of neighborhoods implies $S_1^\alpha = S_2^\beta + p^*$.

Definition 6. The problem Z_l is called locally solvable if it admits a suitable correct local algorithm A_l.

We denote the set of partially marked neighborhoods of points in configurations from the set of precedents by $O_H = \{O(S_\iota^j, (\bar{S}_\iota^{d(\iota)}, \bar{\mu}_\iota^{d(\iota)}))| \bar{S}_\iota^{d(\iota)} \in H, \bar{\mu}_\iota^{d(\iota)} \in H\}$.

Theorem 3 (solvability local criterion). *The problem is locally solvable if and only if the set O_H of partially marked neighborhoods corresponding to the set of precedents H is a consistent set.*

Definition 7. The problem Z_l is called locally regular if it is locally solvable for any suitable partial markings $\bar{\mu}_\iota^{d(\iota)} \in M_\Delta^{d(\iota)}$ of all configurations $\bar{S}_\iota^{d(\iota)}$ from H.

Theorem 4 (regularity local criterion). *A locally solvable problem is locally regular if and only if, for any subset of shift-equivalent neighborhoods of points from the set of precedents, the marking rules imply the existence and uniqueness of a suitable marking.*

Note that the notions introduced and results obtained in this paper make it possible to solve the practically important problem of searching for a minimal system of neighborhoods for a given set of precedents H.

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