Pergamon

Comp Maths Math Phys, Vol. 36, No. 8, pp 1161–1167, 1996

0965–5425/96 $24.00 + 00

PII: S0965-5425(96)00135-4

ALGEBRAIC-LOGIC SYNTHESIS OF CORRECT RECOGNITION PROCEDURES BASED ON ELEMENTARY ALGORITHMS†

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(*Received* 29 *May* 1995)

The construction of minimally complex correct recognition algorithms is investigated. Methods akin to those used when synthesizing logical recognition procedures are proposed. The intention is to extend the range of application of algebraic-logic recognition methods to problems of image and signal processing.

The construction of correct algorithms (by which we mean those that give exact results on the teaching material) by algebraic methods [1–3] reduces essentially to solving the following problems:

(1) the choice of a heuristic data-processing model of the algorithms (for example, a version of the model of algorithms for the calculation of estimates);

(2) the choice of a family of correction operations, which might in a special case even be an algebra (examples of such families are the set of polynomials, monotone Boolean functions, functions with a bounded derivative, etc.);

(3) within the framework of the chosen heuristic data-processing model, the construction of a set of basic algorithms for an existing (given) set of precedents;

(4) the construction of a corrector (an operation which, when applied to the basic algorithms constructed in Step (3), yields a correct algorithm).

Algorithms obtained in this way typically have a complicated internal structure, as reflected in the large number of computer operations required for each object to be recognized. This is not an issue in such areas as medical diagnostics, geological prospecting etc., where the algorithms are applied a relatively small number of times. In image or signal processing, however, the number of calculations involved in a single application of a correct algorithm is an important factor in determining whether practical use can be made of the results.

The present paper is an initial investigation of methods of constructing correct algorithms which are essentially the simplest, in some sense. The basis of the solution technique is modelled on that used in the construction of logical recognition procedures [4–8].

1. BASIC DEFINITIONS AND STATEMENT OF THE PROBLEM

The recognition problem is to be solved in the case where the objects have been described and the initial data about classes have been assigned in the standard way [1].

Consider a certain set of admissible objects M, of each of which it is known that it belongs to one of l subsets (classes) $K_1, \ldots, K_l$. The initial data I_0 (given in the form of a table T) is a sample of a power of m objects from M (the teaching sample). Each row of T is the description of one of the objects in a system

† *Zh Vychisl. Mat. Mat. Fiz.* Vol. 36, No. 8, pp. 215–223, 1996.
Financial support for this research was provided by the Russian Foundation for Basic Research (Project No. 93-01-00457).

of integer indices $\{x_1, \ldots, x_n\}$. Of each object of the teaching sample it is known to which of the classes $K_1, \ldots, K_l$ it belongs.

The recognition problem involves constructing an algorithm which uses the data I_0 and a description S in the system of indices $\{x_1, \ldots, x_n\}$ of an arbitrary object of M (of which, generally speaking, it is not known to which of the classes $K_1, \ldots, K_l$ it belongs) to calculate the values of the properties $S \in K$, $S \notin K$, where $K \in \{K_1, \ldots, K_l\}$.

We will fix the pair $(w, \bar{a})$, where $w \subseteq \{1, 2, \ldots, n\}$, and $w = \{j_1, \ldots, j_r\}, j_1 < \ldots < j_r$, $\bar{a}$ is an ordered set of integers $\{a_1, \ldots, a_r\}$. An elementary recognition algorithm (ERA) is a predicate $P_{(w, \bar{a})}(S)$, defined on descriptions S of objects of M in the system of indices $\{x_1, \ldots, x_n\}$ and taking the value 1 if, and only if, $a_{j_i} = b_i$ with $i + 1, 2, \ldots, r$ for the row $S = (b_1, \ldots, b_n)$.

The set of ERA

$$\mathfrak{U} = \{P^{(1)}(S), \ldots, P^{(q)}(S)\} \tag{1.1}$$

is called a (monotone) correct set of ERA for the class K if there is a (monotone) algebraic logic function F_K of q variables such that $F_K(P^{(1)}(S'), \ldots, P^{(q)}(S')) = 1$ for any row S' of table T which describes an object of class K and $F_K(P^{(1)}(S''), \ldots, P^{(q)}(S'')) = 0$ for any row S'' which describes an object of any other class. The function F_K will be called a (monotone) corrector for the class K.

The following assertion is obvious. The set of ERA of the form (1.1) is a monotone correct set for the class K if, and only if, for any two rows S' and S'' of table T such that $S' \in K$, $S'' \,\overline{\in}\, K$, i in $\{1, 2, \ldots, q\}$ can be found such that

$$P^{(i)}(S') = 1 \text{ and } P^{(i)}(S'') = 0. \tag{1.2}$$

The monotonicity requirement can be removed if condition (1.2) in the last assertion is replaced by the condition

$$P^{(i)}(S') \neq P^{(i)}(S'').$$

Let $\mathfrak{U}$ be a correct set of ERA for the class K. The set $\mathfrak{U}$ will be called a dead-end set if the condition $\mathfrak{U}' \subset \mathfrak{U}$ implies that the set of ERA $\mathfrak{U}'$ is not correct for K. The set $\mathfrak{U}$ will be called minimal if there is no correct set of ERA for K of shorter length.

Let L be any Boolean matrix. The set of columns H of the matrix L will be called a covering if there is 1 at the intersection of each row of L with at least one of the columns in H (that is, a sub-matrix of L formed by columns of H contains no row of the form $(0, \ldots, 0)$). A dead-end covering is a covering which has no proper subset which is also a covering. For a dead-end covering, a sub-matrix of L formed by columns of the set H will contain each of the rows $(1, 0, 0, \ldots, 0, 0)$, $(0, 1, 0, \ldots, 0, 0)$, $\ldots$, $(0, 0, 0, \ldots, 0, 1)$, that is, the set of columns H will contain a sub-matrix, each row and each column of which contains exactly one element equal to 1. A sub-matrix of this kind will be called an identity sub-matrix. A covering with the minimum number of columns will be called minimal.

We shall show that problems of constructing dead-end and minimal correctors reduce, respectively, to the construction of dead-end and minimal coverings for a Boolean matrix constructed from the table T in a special way.

Case 1 (the construction of monotone correct sets of ERA). Let W denote the totality of all possible subsets of the set $\{1, 2, \ldots, n\}$. For each w of W, $w = \{j_1, \ldots, j_r\}$, we will write out all the possible ERA for the class K generated by the columns of table T with numbers $j_1, \ldots, j_r$, that is, all the different sub-descriptions of objects of K that are generated by those columns. Suppose that this is the set of ERA $\mathfrak{U}_w$. Consider the set $\mathfrak{U}_K$ of all ERA for K:

$$\mathfrak{U}_K = \bigcup_{w \in W} \mathfrak{U}_w.$$

Suppose $\mathfrak{U}_K = \{P^{(1)}(S), \ldots, P^{(n')}(S)\}$. We will set the pair of rows S' and S'' of T in correspondence with the row $\mathfrak{B}(S', S'') = (c_1, \ldots, c_{n'})$, in which

$$c_j = \begin{cases} 1, \text{ if } P^{(j)}(S') = 1 \text{ and } P^{(j)}(S'') = 0, \\ 0 \text{ otherwise}, \; j = 1, 2, \ldots n'. \end{cases}$$

We now form a Boolean matrix L_K from all the rows $\mathfrak{B}(S', S'')$ such that $S' \in K$, $S'' \,\overline{\in}\, K$. By construction, each column in L_K corresponds to a certain ERA for K.

It is easy to see that the set of ERA is a monotone dead-end (minimal) correct set for K if, and only if, the corresponding set of columns of the matrix L_K is a dead-end (minimal) covering.

Case 2 (the construction of non-monotone correct sets of ERA). In this case the set of all ERA $\mathfrak{U}'_K$ is formed both by sub-descriptions of objects of K, and by sub-descriptions of objects of other classes. Let $\mathfrak{U}'_K=\{P^{(1)}(S), \ldots, P^{(n'')}(S)\}$. The pair of rows S' and S'' of T are associated with the row $\mathfrak{D}(S', S'')=(d_1, \ldots, d_{n''})$, in which

$$d_j=\begin{cases}1, \text{ if } P^{(j)}(S')\neq P^{(j)}(S''),\\ 0 \text{ otherwise}, \ j=1,2,\ldots,n''.\end{cases}$$

We now form a Boolean matrix from all the rows $\mathfrak{D}(S', S'')$ such that $S'\in K$, $S''\overline{\in} K$. The reasoning is then the same as in Case 1. Only monotone correctors will be considered below.

As a rule, the number of ERA is large even for small problems, and the procedure for constructing a minimal corrector set using the matrix L_K uses a considerable amount of computer resources. The problem thus arises of developing efficient methods for the algebraic-logic correction of ERA in both the general case and in the following cases of practical importance:

(1) the ERA is generated by the pair (j, a), where j is an element of the set $\{1, 2, \ldots, n\}$, and a is one of the admissible values of the index numbered j (note that if $w=\{j_1, \ldots, j_r\}$, $\bar{a}=\{a_1, \ldots, a_r\}$, then $P_{(w,\bar{a})}(S)=P_{(j_1,a_1)}(S)\,\&\ldots\&\,P_{(j_r,a_r)}(S)$);

(2) under the conditions of problem (1), the ERA is defined as the predicate $P_{(w,a)}(S)$, which takes the value 1 if, and only if, the value of the index numbered j for an object S is not less than a.

In each case both accurate and approximate methods are required, as the latter might be necessary to improve the rate of solution.

The procedures normally used for the exact construction of a minimal covering of a Boolean matrix involve an exhaustive search of its coverings.

The search for a minimal covering can also be performed by the examination of dead-end coverings.

Various exact and approximate methods have now been devised for constructing the set of all dead-end coverings of a Boolean matrix. These algorithms are employed in such logical recognition procedures as testing, voting with respect to dead-end representative sets and others. The algorithm of [7, 8], by which the set of all dead-end coverings of a Boolean matrix with limiting minimum complexity is almost always constructed, is especially interesting. It involves finding sets of columns satisfying the dead-end property (in fact all the identity sub-matrices of a given Boolean matrix are constructed). In [7, 8] we give the conditions under which the number of identity sub-matrices is almost always equal in the limit to the number of dead-end coverings and, therefore, the set of columns containing an identity sub-matrix is almost always a covering. We have used this algorithm for the efficient solution of the problem of the determinate (exact) and stochastic (approximate) construction of blind trials and dead-end representative sets. We have also constructed the corresponding recognition procedures. These procedures have been implemented in a number of software packages developed at the Computing Centre of the Russian Academy of Sciences.

2. THE CONSTRUCTION OF A MONOTONE CORRECTOR IN THE CASE OF BINARY DATA

Let T be a binary table and an ERA be generated by the pair (j, a), where j is an element of the set $\{1, 2, \ldots, n\}$, and a is an admissible value of the index x_j.

We shall assume that the first m_1 rows of T are descriptions of objects in the class K, while the next $m_2=m-m_1$ rows describe objects in other classes. Let T_K and $T_{\bar{K}}$ denote the sub-tables of the table T formed by the first m_1 and next m_2 rows, respectively.

Let $i\in\{1, 2, \ldots, m_1\}$, and S_i be row i of table T. We will denote by L^i_K a sub-matrix of the matrix L_K formed by rows of the form $\mathfrak{B}(S_i, S)$, where $S\overline{\in} K$.

It is easy to see that if the jth element of row S_i is equal to 0, then the column in L^i_K corresponding to the ERA $(j, 0)$ is the same as the jth column of table $T_{\bar{K}}$, and the column corresponding to the ERA $(j, 1)$ is zero. And, on the other hand, if the jth element of row S_i is equal to 1, the column in L^i_K corresponding to the ERA $(j, 0)$ is zero, and the column corresponding to the ERA $(j, 1)$ is opposite the jth row of table $T_{\bar{K}}$.

The ERA (j, a) will be called empty for S_i if the jth element of row S_i is not equal to a.

We now remove from L_K^i every zero column corresponding to empty ERA, and obtain a Boolean matrix M_K^i of dimensions $m_2 \times n$.

Let $S_i = (a_{i1}, \ldots, a_{in})$. Then the matrix M_K^i will obviously consist of all rows of the form $(a_{i1} \oplus b_1, \ldots, a_{in} \oplus b_n)$, where $(b_1, \ldots, b_n)$ is a row of table $T_{\bar{K}}$, and $\oplus$ denotes addition mod 2. By construction, each unit element e in M_K^i of the form $a_{ij} \oplus b_j$ corresponds to an ERA for the class K of the form (j, a_{ij}), which we will denote by $R(e)$.

In each sub-matrix $L_K^1, \ldots, L_K^{m_1}$ of the matrix L_K, we now remove all zero columns corresponding to empty ERA. This transforms the matrix L_K to the matrix M_K of dimensions $m_1 m_2 \times n$.

In this case (cf. Section 1) the problem of constructing the set $\mathfrak{M}(K)$ of all monotone dead-end correct sets of ERA for the class K reduces to constructing the set of all dead-end coverings for the matrix L_K of dimensions $m_1 m_2 \times 2n$. We will show that this latter problem can be reduced to finding special sets of unit elements of the matrix M_K constructed above. This will lead to a considerable reduction in the amount of calculation, since there are half as many columns in M_K as in L_K. A solution is found using a modification of the algorithm of [7, 8] for constructing the set of all dead-end coverings.

Two unit elements of the matrix L_K which are situated in row i_1 and i_2 and column j_1 and j_2, respectively, are said to be compatible if $i_1 \neq i_2, j_1 \neq j_2$ and the sub-matrix of L_K formed by rows i_1 and i_2 and columns j_1 and j_2 is an identity sub-matrix, that is, it consists of the two rows (1, 0) and (0, 1). The set E of r unit elements of the matrix L_K is said to be compatible if either: (1) $r = 1$, or (2) $r > 1$ and any two elements in E are compatible.

It is easy to see that the set H of r columns of the matrix L_K is a dead-end covering if, and only if, first, r compatible elements can be found in L_K which are situated in columns of the set H and, second, H is a covering, that is, there is a 1 at the intersection of each row of the matrix L_K with at least one column of H.

Let $i \in \{1, 2, \ldots, m_1 m_2\}$. Then the number i can be represented uniquely in the form

$$i = (t_1 - 1)m_2 + t_2, \quad t_1 \in \{1, 2, \ldots, m_1\}, \quad t_2 \in \{1, 2, \ldots, m_2\}. \tag{2.1}$$

We will denote the numbers t_1 and t_2 in the representation of the number i (2.1) by $x(i)$ and $y(i)$, respectively.

Two elements of the matrix M_K situated in the same column and in rows i_1 and i_2 will be called equivalent if $x(i_1) \neq x(i_2)$, $y(i_1) = y(i_2)$, that is, $i_1 \equiv i_2 \pmod{m_2}$.

In sub-matrices M_K^p and M_K^q $(p, q \in \{1, 2, \ldots, m_1\})$ of the matrix M_K, let the unit elements e_1 and e_2, respectively, lie in rows i_1 and i_2 and in columns j_1 and j_2. We shall call the elements e_1 and e_2 similar if one of the following conditions is satisfied:

(1) e_1 and e_2 are compatible;

(2) e_1 and e_2 are not compatible, $p \neq q$ and one of the following conditions is satisfied:

(a) e_1 has a zero equivalent element in M_K^q, and e_2 has a zero equivalent element in M_K^p (it could be that $j_1 = j_2$);

(b) $j_1 \neq j_2$ and either there is a 0 on the intersection of row i_2 and column j_1 and the equivalent element to e_2 in M_K^p is zero or, on the other hand, there is 0 at the intersection of row i_1 and column j_2 and the equivalent element of e_1 in M_K^q is zero.

The set Q of r unit elements of the matrix M_K will be called similar if either: (1) $r = 1$, or (2) $r > 1$ and any two elements in Q are similar.

We will denote the set of all similar sets of unit elements in M_K by S_K.

We will denote the set of all columns of the matrix M_K which contain elements of the set Q by $\Omega(Q)$, $Q \in S_K$.

We will now show that the problem of constructing the set $\mathfrak{M}(K)$ of all dead-end coverings for the matrix L_K reduces to the problem of constructing the set of all sets Q of S_K such that the set of columns $\Omega(Q)$ is a covering.

The transformation of the matrix L_K to the matrix M_K described above obviously defines a one-to-one correspondence between the unit elements of these matrices, each unit element e of the matrix L_K corresponding to a unit element $\Pi(e)$ of the matrix M_K.

Theorem 1. Two unit elements e_1 and e_2 are compatible in L_K if, and only if the corresponding unit elements $\Pi(e_1)$ and $\Pi(e_2)$ in M_K are similar.

Proof. In the matrix L_K, let the unit element e_1 of the sub-matrix L_K^p lie in row i_1 and the column corresponding to ERA (j_1, a_1), and the unit element e_2 of the sub-matrix L_K^q in row i_2 and the column corresponding to ERA (j_2, a_2).

I. Let the elements e_1 and e_2 be compatible in L_K. Let us consider the possible cases.

If $p=q$, then obviously $\Pi(e_1)$ and $\Pi(e_2)$ are compatible in M_K and are therefore similar in M_K.

Let $p\neq q$ and the rows S_p and S_q of table T be of the form $(a_{p1}, \ldots, a_{pn})$ and $(a_{q1}, \ldots, a_{qn})$, respectively. Then $a_{pj_1}=a_1$, $a_{qj_2}=a_2$.

Suppose that

$$a_{qj_1}=a_1,\quad a_{pj_2}=a_2. \tag{2.2}$$

Then the columns corresponding to ERA (j_1, a_1) and (j_2, a_2) are not zero either in L_K^p or in L_K^q, that is, $\Pi(e_1)$ and $\Pi(e_2)$ are compatible in M_K.

But if only one of Eqs (2.2) is satisfied, the first, say, it follows from $a_{pj_1}=a_{qj_1}=a_1$ that the column in L_K^p corresponding to ERA (j_1, a_1) is non-zero and is the same as the same column in L_K^q. It follows from $a_{qj_2}=a_2$, $a_{pj_2}=\bar{a}_2$ that the column in L_K^q corresponding to ERA (j_2, a_2) is opposite the column in L_K^q corresponding to ERA $(j_2, \bar{a}_2)$. Thus $\Pi(e_1)$ and $\Pi(e_2)$ are similar in M_K in accordance with part (b) of the definition of the similarity of two elements.

Finally, if neither equation of (2.2) is satisfied, it follows from the equations

$$a_{pj_1}=a_1,\quad a_{qj_1}=\bar{a}_1,\quad a_{pj_2}=\bar{a}_2,\quad a_{qj_2}=a_2$$

that the column in L_K^p corresponding to ERA (j_1, a_1) is opposite the column in L_K^q corresponding to ERA $(j_1, \bar{a}_1)$. Similarly, the column in L_K^q corresponding to ERA (j_2, a_2) is opposite the column in L_K^p corresponding to ERA $(j_2, \bar{a}_2)$. It follows that $\Pi(e_1)$ and $\Pi(e_2)$ are similar in M_K in accordance with part (a) of the definition of the similarity of two elements.

II. Suppose that elements e_1 and e_2 are not compatible in L_K.

Let e_3 denote the element of the submatrix L_K^q which lies in row i_2 and the column corresponding to ERA (j_1, a_1), and e_4 the element of the sub-matrix L_K^p which lies in row i_1 and the column corresponding to ERA (j_2, a_2). Since e_1 and e_2 are incompatible in L_K, at least one of the two following equations is satisfied: $e_3=1$, $e_4=1$. Let $e_3=1$, for instance. Then the columns which correspond in L_K^p and L_K^q to ERA (j_1, a_1) are the same and, therefore, the element in M_K equivalent to $\Pi(e_1)$ is equal to 1 and the element $\Pi(e_3)$ is equal to 1. It follows that $\Pi(e_1)$ and $\Pi(e_2)$ are not similar in M_K. This proves the theorem.

Let Q be the set of elements of the matrix M_K, $Q=\{e_1, \ldots, e_r\}$. It follows from Theorem 1 that the set ERA $\{R(e_1), \ldots, R(e_r)\}$ belongs to $\mathfrak{M}(K)$ if, and only if, first, $Q\in S_K$ and, second, the set of columns $\Omega(Q)$ is a covering for M_K.

We will now describe an algorithm $\mathfrak{A}$ for constructing the set $\mathfrak{M}(K)$ by finding sets from S_K.

Let E_K denote the set of all unit elements in M_K.

We will give the element e in row i and column j of the matrix M_K the number $N[e]=(j-1)m_1m_2+i$. We will denote the elements with the smallest and largest numbers in any set E, $E\subseteq E_K$ by $q_1(E)$ and $q_2(E)$, respectively.

We introduce a linear ordering on the set S_K. For each set Q, $Q\in S_K$ and $Q\neq\{q_2(E_K)\}$, we define the next set $\square Q$ of S_K.

Suppose that $Q=\{e_1, \ldots, e_r\}$ and $N[e_{u+1}]>N[e_u]$ for $u=1, 2, \ldots, r-1$.

Let $u\in\{1, 2, \ldots, r\}$. We will denote the set of all elements in E_K whose numbers are greater than $N[e_u]$ by E_u. We pick out a subset G_u in E_u, the elements which are similar to each of the elements $e_1, \ldots, e_u$.

There are the following possible cases, for each of which we will indicate $\square Q$:

(1) $G_r\neq\varnothing$, in which case $\square Q=Q\cup\{q_1(G_r)\}$;

(2) $G_r=\varnothing$;

(a) $r=1$, in which case $\square Q=\{q_1(E_1)\}$;

(b) $r>1$ and $G_{r-1}\cap E_r\neq\varnothing$, then $\square Q=(Q\setminus\{e_r\})\cup\{q_1(G_{r-1}\cap E_r)\}$;

(c) $r>1$ and $G_{r-1}\cap E_r=\varnothing$, then if $r=2$, we have $\square Q=\{q_1(E_1)\}$ and if $r>2$ we have $\square Q=(Q\setminus\{e_{r-1}, e_r\})\cup\{q_1(G_{r-2}\cap E_{r-1})\}$.

We note that $G=G_{r-2}\cap E_{r-1}\neq\varnothing$ for $r>2$ since $e_r\in G$.

For $\{R(e_1), \ldots, R(e_r)\}$ to belong to $\mathfrak{M}(K)$ obviously the condition $G_r=\varnothing$ must be satisfied. Sufficient conditions are: $G_r=\varnothing$ and the set of columns $\Omega(Q)$ of the matrix M_K is a covering.

The set $\{e_1, \ldots, e_r\}$ will be called an upper set if there is no set $\{e'_1, \ldots, e'_r\}$ belonging to S_K with the following properties:

(1) the element e'_u lies in the same column of M_K as the element e_u, $u = 1, 2, \ldots, r$;

(2) for at least one u of $\{1, 2, \ldots, r\}$ we have $N[e'_u] < N[e_u]$, $R(e'_u) = R(e_u)$ (the condition $R(e'_u) = R(e_u)$ means that either the elements e'_u and e_u belong to the same sub-matrix $M_K^{p_u}$, $p_u \in \{1, 2, \ldots, m_1\}$ or these elements are equivalent in M_K).

Let S_K^* denote the set of all upper sets in S_K.

The algorithm $\mathfrak{A}$ constructs the set $\mathfrak{M}(K)$ in $|S_K|$ steps. On step i $\mathfrak{A}$ selects in M_K a similar set of elements $\mathfrak{A}[i, M_K] = \{e_1, \ldots, e_r\}$. In order to eliminate repeats, it does this by checking that $\mathfrak{A}[i, M_K] \in S_K^*$ for $i \geqslant 2$ and $G_i = \varnothing$. If this condition is satisfied and the set of columns $\Omega(Q)$ of the matrix M_K is a covering, the set of ERA $\{R(e_1), \ldots, R(e_r)\}$ is included in the required set of dead-end correct sets of ERA. If not, the set of dead-end correct sets constructed in the preceding steps is left unchanged.

The choice of similar sets is based on the following rules:

(1) $\mathfrak{A}[1, M_K] = \{q_1(E_K)\}$;

(2) if $\mathfrak{A}[i, M_K] \neq \{q_2(E_K)\}$, then $\mathfrak{A}[i+1, M_K] = \square\mathfrak{A}[i, M_K]$;

(3) if $\mathfrak{A}[i, M_K] = \{q_2(E_K)\}$, algorithm $\mathfrak{A}$ completes its operation.

Let $Q \in S_K$. Consider the sub-matrix $M(Q)$ formed by rows and columns which contain elements of the set Q. It is easy to see that each row of the submatrix $M(Q)$ contains exactly one element equal to 1, and that each column has at most two elements equal to 1. Note that if some column of that sub-matrix has two unit elements and they belong to M_K^i and M_K^j, then $i \neq j$. We will call $M(Q)$ a quasi-identity sub-matrix.

Quasi-identity sub-matrices of the matrix M_K can be found by finding its identity sub-matrices, using the algorithm of [7, 8] directly for that purpose.

The exhaustive search involved in the construction of dead-end coverings can be shortened by removing "including" rows from the Boolean matrix.

Let two rows of the Boolean matrix L be of the form $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_n)$, where $c_j \geqslant b_j$ for $j = 1, 2, \ldots, n$. The second row will be said to include the first, and this will be denoted by $(b_1, \ldots, b_n) < (c_1, \ldots, c_n)$, if $c_j > b_j$ for at least one j from $\{1, 2, \ldots, n\}$. The removal of including rows from L will not alter the set of its coverings.

Suppose that there is a 1 at the intersection of rows i_1 and i_2 with column j of the matrix M_K. Let Λ_{i_1} and Λ_{i_2} denote the rows of matrix L_K numbered i_1 and i_2, respectively. We then have the following theorem.

Theorem 2. The inclusion $\Lambda_{i_1} < \Lambda_{i_2}$ is satisfied if, and only if, the following two conditions hold:

(1) row i_2 in the matrix M_K includes row i_1;

(2) the elements at the intersection of rows $x(i_1)$ and $x(i_2)$ in the matrix T_K with column j are equal.

Proof. Let the elements at the intersection of the rows $x(i_1)$ and $x(i_2)$ in T_K with column j be a and a' respectively.

If $a = a'$, the elements at the intersection of rows S' and S'' of the matrix T_K, numbered $y(i_1)$ and $y(i_2)$, respectively, with column j are $\bar{a}$, and in the matrix L_K the elements at the intersection of the column corresponding to ERA (j, a) with rows Λ_{i_1} and Λ_{i_2} are 1.

The conditions stated in the theorem are therefore sufficient.

The necessity of condition (1) is obvious. The necessity of condition (2) is also easy to see. For if $a' = \bar{a}$, the elements at the intersection of rows S' and S'' of matrix T_K with column j are $\bar{a}$ and a. Hence, there are 1 and 0 respectively at the intersection of the column of matrix L_K corresponding to ERA (j, a) and rows Λ_{i_1} and Λ_{i_2} and, on the other hand, there are 0 and 1, respectively, at the intersection of the column corresponding to ERA $(j, \bar{a})$ and rows Λ_{i_1} and Λ_{i_2}. Hence it follows that row Λ_{i_2} in L_K does not include row Λ_{i_1}. This proves the theorem.

Corollary 1. In the matrix M_K, let row i_2 include row i_1. Then if $x(i_1) = x(i_2)$ we have $\Lambda_{i_1} < \Lambda_{i_2}$.

Corollary 2. In the matrix M_K, let row i_2 include row i_1. Then if $x(i_1) \neq x(i_2)$ and $y(i_1) = y(i_2)$, that is, rows i_1 and i_2 of the matrix M_K are equivalent, we have $\Lambda_{i_1} < \Lambda_{i_2}$.

Note that this method of constructing the set of all dead-end correct sets of ERA can also be used in the more general case of integer-valued data. The exhaustive search will be even shorter in that case.

1. ZHUI
Cyber
2. ZHUI
21–2
3. RUD.
Class
4. DMIT
object
5. VAIN
82–9
6. BASK
suppo
7. DYUI
165–
8. DYUI
Class

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