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On the Number of Irreducible Coverings of an Integer Matrix

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Abstract—The metric (quantitative) properties of the set of coverings of an integer matrix are examined. An asymptotic estimate for the logarithm of the typical number of irredundant σ-coverings is obtained in the case when the number of rows in the matrix is not smaller than the number of its columns. As a consequence, a similar estimate is derived for the number of maximal conjunctions of a Boolean function of n variables with the number of zeros no less than n.

Keywords: discrete recognition procedures, irredundant covering of an integer matrix, metric properties of a set of coverings, metric properties of disjunctive normal forms.

The importance of the problem under study is motivated by the necessity of constructing efficient implementations for discrete (logical) recognition and classification procedures, which require considerable computational costs. A related problem is that of estimating the number of maximal conjunctions of Boolean functions.

Let M_{mn}^k be the set of all $m \times n$ matrices with elements from $\{0, 1, \ldots, k-1\}$, where $k \geq 2$; and let E_k^r with $k \geq 2$ and $r \leq n$ be the set of all tuples of the form $(\sigma_1, \ldots, \sigma_r)$, where $\sigma_i \in \{0, 1, \ldots, k-1\}$.

Let $L \in M_{mn}^k$ and $\sigma \in E_k^r$. Recall that a σ-*covering* of L is a set H of columns in L such that the submatrix L^H of L formed of the columns of H does not contain σ (see [1, 2]). An *irredundant* σ-*covering* of L is a set H of columns that is a covering such that L^H contains (up to row permutation) a submatrix of the form

$$\begin{bmatrix} \beta_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_{r-1} & \sigma_r \\ \sigma_1 & \beta_2 & \sigma_3 & \ldots & \sigma_{r-1} & \sigma_r \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_{r-1} & \beta_r \end{bmatrix},$$

where $\beta_p \neq \sigma_p$ for $p = 1, 2, \ldots, r$. Such a submatrix is called a σ-submatrix.

The concept of an irredundant $(0, \ldots, 0)$-covering of a Boolean matrix coincides with the well-known concept of an irreducible covering of a Boolean matrix. Note that a $(0, \ldots, 0)$-submatrix of a Boolean matrix is an identity submatrix.

Let $C(L, \sigma)$ be the set of all σ-coverings of L, $S(L, \sigma)$ be the set of all σ-submatrices of L, and $B(L, \sigma)$ be the set of all irredundant σ-coverings of L. Furthermore, let

$$C_r(L) = \bigcup_{\sigma \in E_k^r} C(L, \sigma), \quad S_r(L) = \bigcup_{\sigma \in E_k^r} S(L, \sigma), \quad B_r(L) = \bigcup_{\sigma \in E_k^r} B(L, \sigma),$$

$$C(L) = \bigcup_{r=1}^{n} C_r(L), \quad S(L) = \bigcup_{r=1}^{n} S_r(L), \quad B(L) = \bigcup_{r=1}^{n} B_r(L),$$

and $|A|$ denote the cardinality of a set A.

It was shown in [1, 2] that the asymptotic behavior of $|B(L)|$ coincides with that of $|S(L)|$ for almost all matrices in M_{mn}^k if $n \longrightarrow \infty$ and $m^\alpha \leq n \leq k^{m^\beta}$, where $\alpha > 1$ and $\beta < 1$. Moreover, the number of coverings from $B(L)$ is almost always less in order than the number of coverings from $C(L)$. Based on these results, an algorithm was designed in [3] that finds all the irredundant $(0, \ldots, 0)$-coverings of a Boolean matrix with a polynomial time delay. This algorithm is a modification of that constructed in [1], which solves the same problem approximately by a polynomial search for all identity submatrices of the original matrix. A drawback of both algorithms is that they involve repeatedly constructed irreducible coverings. In the case under consideration, the number of steps in each algorithm almost always (for almost all $m \times n$ Boolean matrices) asymptotically coincides with the number of irredundant (0, …, 0)-coverings as $n \longrightarrow \infty$. Theoretical and experimental estimates are obtained for the complexity of the exact and approximate algorithms for other cases. The algorithms can easily be modified to the case of constructing irredundant coverings of an integer matrix.

Less thoroughly studied is the opposite case, namely, $n^{\alpha} \le m \le k^{n^{\beta}}$, where $\alpha > 1$, $\beta < 1$, and $n \longrightarrow \infty$. An asymptotics of the typical number of submatrices from $S(L)$ was obtained in this case in [4]. Additionally, in [4], an asymptotics of the typical number of coverings from $C(L)$ was obtained in a virtually general case, namely, for $m \le k^{n^{\beta}}$ with $\beta < 1$. For $n^{\alpha} \le m \le k^{n^{\beta}}$, where $\alpha > 1$ and $\beta < 1/2$, it was established that $|S(L)|$ is almost always greater in order than $|B(L)|$ as $n \longrightarrow \infty$.

Let $r_1 = [\log_k m - \log_k \ln \log_k m] - 1$.

Theorem 1. *If* $n \le m \le k^{n^{\beta}}$, *where* $\beta < 1/2$, *then, for almost all matrices* L *in* M^k_{mn},

$$\log_k |B(L)| \approx \log_k C_n^{r_1} + r_1$$

as $n \longrightarrow \infty$.

The proof of this theorem is based on Lemma 1–7 given below.

Suppose that $M^k_{mn} = \{L\}$ is the space of elementary events, with every event L occurring with a probability of $1/|M^k_{mn}|$. The expectation of a random variable $X(L)$ defined on M^k_{mn} is denoted by $\mathbf{M}X(L)$ and its variance by $\mathbf{D}X(L)$.

The proofs of Lemmas 3, 5, and 6 make use of the Chebyshev inequality formulated in Proposition 1, easy-to-prove Proposition 2, and Proposition 3, which is a straightforward consequence of Proposition 1.

Proposition 1. *Let* $\theta > 0$ *and* $\Delta_\theta(n)$ *be the fraction of matrices* L *in* M^k_{mn} *for which* $|X(L) - \mathbf{M}X(L)| \ge \theta$. *Then*, $\Delta_\theta(n) \le \mathbf{D}X(L)/\theta^2$.

Proposition 2. *Let* $\theta > 0$ *and* $\nu_\theta(n)$ *be the fraction of matrices* L *in* M^k_{mn} *for which* $X(L) \ge \theta \mathbf{M}X(L)$. *Then*, $\nu_\theta(n) \le 1/\theta$.

Proposition 3. *If* $\mathbf{M}X(L) \longrightarrow 0$ *as* $n \longrightarrow 0$, *then* $X(L) = 0$ *for almost all matrices* L *in* M^k_{mn}.

On M^k_{mn}, consider the random variables $\xi_r(L) = |C_r(L)|$ and $\eta_r(L) = |B_r(L)|$. It is easy to calculate that

$$\mathbf{M}\xi_r(L) = C_n^r k^r (k^r - 1)^m k^{-mr} = C_n^r k^r (1 - k^{-r})^m,$$

$$\mathbf{M}\eta_r(L) = C_n^r k^r \sum_{i=0}^{r} (-1)^i C_r^i (k^r - [i(k-1)+1]^m k^{-mr} = C_n^r k^r \sum_{i=0}^{r} (-1)^i C_r^i \{1 - [i(k-1)+1] k^{-r}\}^m = C_n^r k^r E.$$

Lemma 1 (see [6]). *Let*

$$S(r, t) = \sum_{i=0}^{r} (-1)^{r-i} C_r^i [i(k-1)+1]^t.$$

Then,

(i) $S(r, t) = 0$ *for* $t < r(k-1)$;

(ii) $S(r, t) \le (r+1)^t$ *for* $t \ge r(k-1)$.

In the case $k = 2$, Lemma 1 was proved in [6]. Its proof in the general case is entirely similar.

Let $r_2 =]\log_k m + 3[$.

Lemma 2. *For all* $r > r_2$,

$$\mathbf{M}\eta_r(L) \le k^{r+1} C_n^r \left(\frac{m}{k^{r-3}}\right)^{r(k-1)}.$$

Proof. Applying Lemma 1, we have

$$E \le \sum_{i=0}^{r}(-1)^i C_r^i \sum_{t=0}^{m}(-1)^t C_m^t k^{-rt}[i(k-1)+1]^t \le \sum_{t=0}^{m} C_m^t k^{-rt}\sum_{i=0}^{r}(-1)^{r-i}C_r^i[i(k-1)+1]^t(-1)^{t-r}$$

$$\le \sum_{t \ge r(k-1)}^{m} C_m^t k^{-rt}[r(k-1)+1]^t.$$

Let $a_t = C_m^t k^{-rt}[r(k-1)+1]^t$, where $t \ge r(k-1)$ and $r > r_2$. Consider

$$\frac{a_{t+1}}{a_t} = \frac{(m-t)[r(k-1)+1]}{(t+1)k^r} \le mk^{-1} < 1/8.$$

Then, we have

$$E \le \frac{8}{7}C_m^{r(k-1)}k^{-r^2(k-1)}[r(k-1)+1]^{r(k-1)} < k^{3r(k-1)+1}(m/k^r)^{r(k-1)}.$$

The lemma is proved.

In what follows, the notation $a_n \le_n b_n$, $n \longrightarrow \infty$ means that $\lim(a_n/b_n) \le 1$ as $n \longrightarrow \infty$.

Lemma 3. *If* $n \le m \le k^{n^\beta}$, *where* $\beta < 1$, *then, for almost all matrices* $L \in M_{mn}^k$,

$$\log_k|B(L)| \le_n \log_k C_n^{r_1} + r_1$$

as $n \longrightarrow \infty$.

Proof. By Lemma 2, we have

$$S = \mathbf{M}|B(L)| = \sum_{r=1}^{n}\mathbf{M}\eta_r(L) \le \sum_{r<r_1}\mathbf{M}\xi_r(L) + \sum_{r_1 \le r \le r_2} C_n^r k^r + k\sum_{r>r_2} C_n^r k^r\left(\frac{m}{k^{r-3}}\right)^{r(k-1)}.$$

Note that $\mathbf{M}\xi_{r-1}(L) = o(\mathbf{M}\xi_r(L))$ as $n \longrightarrow \infty$, where $r < r_1$, $m \le k^{n^\beta}$, and $\beta < 1$.

Let

$$Q = \max\left\{1, \left(\frac{n-r_1}{r_1}k\right)^{r_2-r_1}\right\}.$$

Then, for every $r \ge r_1$, it holds that $C_n^r k^r \le QC_n^{r_1}k^{r_1}$, and for $m \ge n$, we obtain

$$\sum_{r>r_2} C_n^r k^r\left(\frac{m}{k^{r-3}}\right)^{r(k-1)} \le QC_n^{r_1}k^{r_1}.$$

Consequently, $S \le_n (r_2 - r_1 + 1)QC_n^{r_1}k^{r_1}$.

By Proposition 2, the fraction of matrices L for which $|B(L)| \ge \dfrac{\log_k n}{r_2 - r_1 + 1}S$ does not exceed $\dfrac{r_2 - r_1 + 1}{\log_k n}$.

Therefore, for almost all matrices L in M_{mn}^k, we have $|B(L)| \le_n (Q\log_k n)C_n^{r_1}k^{r_1}$. Taking into account $\log_k(Q\log_k n) = o(\log_k C_n^{r_1})$, we obtain the required assertion.

Lemma 4 (see [5]). *If* $m \le k^{n^\beta}$, *where* $\beta < 1/2$, *then, for almost all matrices* L *in* M_{mn}^k,

$$\mathbf{D}\xi_{r_1}(L) \le \delta(n)[\mathbf{M}\xi_{r_1}(L)]^2,$$

where $\delta(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Lemma 4 and the Chebyshev inequality (see Proposition 1) imply the following result.

Lemma 5. *If* $m \le k^{n^\beta}$, *where* $\beta < 1/2$, *then, for almost all matrices L in* M_{mn}^k,

$$\left|\xi_{r_1}(L) - \mathbf{M}\xi_{r_1}(L)\right| \le \delta(n)\mathbf{M}\xi_{r_1}(L),$$

where $\delta(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Lemma 6. *Let* $\mu_r(L)$ *be the number of rank-r coverings of a matrix* $L \in M_{mn}^k$ *that are not irredundant. Then, the fraction of matrices satisfying* $\mu_{r_1}(L) \ge \mathbf{M}\xi_{r_1}(L)\log_k^{4/3-k}m$ *tends to zero as n increases.*

Proof. Let us estimate $\mathbf{M}\mu_r(L)$ from above:

$$\mathbf{M}\mu_r(L) = \mathbf{M}\xi_r(L) - \mathbf{M}\eta_r(L) = C_n^r k^r \sum_{i=1}^{r} (-1)^{i-1} C_r^i [k^r - (i+1)]^m k^{-mr} \le r(1-k^{-r})^m \mathbf{M}\xi_r(L).$$

Setting $r = r_1$, we obtain

$$\mathbf{M}\mu_{r_1}(L) \le \frac{r_1}{\log_k^k m}\mathbf{M}\xi_{r_1}(L) \le \frac{1}{\log_k^{k-1} m}\mathbf{M}\xi_{r_1}(L).$$

Applying Proposition 2 with $\theta = \log_k^{1/3} m$ gives the required result.

Lemma 7. *If* $n \le m \le k^{n^\beta}$, *where* $\beta < 1/2$, *then, for almost all matrices L in* M_{mn}^k,

$$\log_k|B(L)| \ge_n \log_k C_n^{r_1} + r_1$$

as $n \longrightarrow \infty$.

Proof. Obviously, $|B(L)| \ge \xi_{r_1}(L) - \mu_{r_1}(L)$.

Lemmas 5 and 6 imply that

$$\xi_{r_1}(L) \ge [1 - \delta(n)]\mathbf{M}\xi_{r_1}(L), \quad \delta(n) \longrightarrow 0, \quad \mu_{r_1}(L) \le (\log_k m)^{4/3-k}\mathbf{M}\xi_{r_1}(L)$$

for almost all matrices $L \in M_{mn}^k$. Thus, we have

$$|B(L)| \ge C_n^{r_1} k^{r_1}(1 - k^{-r_1})^m [1 - \delta(n) - (\log_k m)^{4/3-k}].$$

Estimating

$$(1 - k^{-r})^m \ge \exp(-2m/k^{r_1}) \ge (\log_k m)^{-2k}.$$

and taking onto account l $\log_k \log_k m = o(\log_k C_n^{r_1})$ yields the required result.

The theorem is an immediate consequence of Lemmas 3 and 7.

Remark. In the proof of Lemma 1, we used the methods proposed in [6] for obtaining asymptotics for the logarithm of the typical number of maximal conjunctions of a partial Boolean function. The problem consider in [6] is somewhat more complicated than that solved here. For this reason, the technical constructions used simplified considerably.

The upper bound for $\log_k|B(L)|$ given in Lemma 3 can be derived in a different manner, namely, by estimating the expectation of the number of submatrices in $S(L)$ with the rank no less than r_1 and estimating the expectation of the number of coverings in $C(L)$ with the length shorter than r_1.

Indeed, let $\zeta_r(L) = |S_r(L)|$, $L \in M_{mn}^k$. It is easy to calculate that

$$\mathbf{M}\zeta_r(L) = C_n^r C_m^r r!(k-1)^r k^{r-r^2}.$$

Since $\mathbf{M}\zeta_{r+1}(L) = o(\mathbf{M}\zeta_r(L))$ for $r \ge r_1$, we have

$$\sum_{r \ge r_1} \mathbf{M}\eta_r(L) \le \sum_{r \ge r_1} \mathbf{M}\zeta_r(L) \approx C_n^{r_1} C_m^{r_1} r_1!(k-1)^{r_1} k^{r_1 - r_1^2}, \quad n \longrightarrow \infty. \tag{1}$$

On the other hand,

$$\sum_{r<r_1} \mathbf{M}\eta_r(L) \le \sum_{r<r_1} \mathbf{M}\xi_r(L) = \sum_{r<r_1} C_n^r k^r (1-k^{-r})^m = o(C_n^{r_1} k^{r_1}), \quad n \longrightarrow \infty. \tag{2}$$

It follows from (1) and (2) that

$$\mathbf{M}|B(L)| \le_n C_n^{r_1} C_m^{r_1} r_1! (k-1)^{r_1} k^{r_1 - r_1^2}, \quad n \longrightarrow \infty.$$

By Proposition 2, $|B(L)| \le \theta \mathbf{M}|B(L)|$ with $\theta = \log_k m$ for almost all matrices L in M^k_{mn}.

Therefore, $\log_k|B(L)| \le n\log_k C_n^{r_1} + r_1$, which was to be proved.

The bound obtained in Theorem 1 can be used to estimate the typical length of the reduced disjunctive normal form (DNF) of a special Boolean function.

Indeed, suppose that $F(x_1, \ldots, x_n)$ is defined on E_k^n, take values in $\{0, 1\}$, and let N^k_{mn} be the set of all such functions. Define

$$x^\sigma = \begin{cases} 1 & \text{if } x = \sigma, \\ 0 & \text{if } x \ne \sigma, \end{cases}$$

where $x, \sigma \in \{0, 1, \ldots, k-1\}$.

The concept of a minterm (elementary conjunction) is introduced in the ordinary way. A *minterm over* $x_1, \ldots, x_n$ is an expression of the form $x_{j_1}^{\sigma_1} \ldots x_{j_r}^{\sigma_r}$, where $x_{j_i} \in \{x_1, \ldots, x_n\}$ for $i = 1, 2, \ldots, r$ and $x_{j_q} \ne x_{j_t}$ for $t, q \in \{1, 2, \ldots, r\}$, $t \ne q$. A minterm is equal to 1 if and only if each of its factor is equal to 1.

Let N_B denote the truth interval of a minterm B. A minterm B is called admissible for F if $N_B \cap A_K \ne \varnothing$ and $N_B \cap B_K = \varnothing$. A minterm B is called maximal for F if it is admissible and there is no admissible minterm B' such that $N_{B'} \supset N_B$.

Propositions 4 and 5 below are obvious.

Proposition 4. *A minterm* $x_{j_1}^{\sigma_1} \ldots x_{j_r}^{\sigma_r}$ *is admissible for F if and only if the set of columns in L indexed by* $j_1, \ldots, j_r$ *is a* $(\sigma_1, \ldots, \sigma_r)$*-covering.*

Proposition 5. *A minterm* $x_{j_1}^{\sigma_1} \ldots x_{j_r}^{\sigma_r}$ *is maximal for F if and only if the set of columns in L indexed by* $j_1, \ldots, j_r$ *is an irredundant* $(\sigma_1, \ldots, \sigma_r)$*-covering.*

It is also easy to prove the following.

Proposition 6. *If* $n \le m$, *where* $m^2 = \bar{o}(k^n)$, *then almost all matrices in* M^k_{mn} *are matrices with pairwise different rows when* $n \longrightarrow \infty$.

Denote by $l(F)$ the length of the reduced DNF of F from N^k_{mn}.

Theorem 1 and Propositions 4–6 imply the following result.

Theorem 2. *If* $n \le m \le k^{n^\beta}$, *where* $\beta < 1/2$, *then*

$$\log_k l(F) \approx \log_k C_n^{r_1} + r_1$$

for almost all functions F in N^k_{mn} *as* $n \longrightarrow \infty$.

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REFERENCES

1. E. V. Djukova, “On the Complexity of Implementation of Some Recognition Procedures,” Zh. Vychisl. Mat. Mat. Fiz. **27**, 114–127 (1987).
2. E. V. Djukova and Yu. I. Zhuravlev, “Discrete Analysis of Feature Descriptions in Recognition Problems of High Dimensionality,” Zh. Vychisl. Mat. Mat. Fiz. **40** (8), 1264–1278 (2000) [Comput. Math. Math. Phys. **40**, 1214–1227 (2000)].
3. E. V. Djukova, “On the Implementation Complexity of Discrete (Logical) Recognition Procedures,” Zh. Vychisl. Mat. Mat. Fiz. **44**, 562–572 (2004) [Comput. Math. Math. Phys. **44**, 532–541 (2004)].
4. E. V. Djukova and N. V. Peskov, “Search for Informative Fragments in Descriptions of Objects in Discrete Recognition Procedures,” Zh. Vychisl. Mat. Mat. Fiz. **42**, 741–753 (2002) [Comput. Math. Math. Phys. **42**, 711–723 (2002)].
5. E. V. Djukova and A. S. Inyakin, “Classification Procedures Based on the Construction of Class Coverings,” Zh. Vychisl. Mat. Mat. Fiz. **43**, 1884–1895 (2003) [Comput. Math. Math. Phys. **43**, 1812–1822 (2003)].
6. A. A. Sapozhenko, “An Estimate of the Length and Number of Irredundant Disjunctive Normal Forms for Almost All Incompletely Defined Boolean Functions,” Mat. Zametki **28** (2), 279–299 (1980).