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NONHOLONOMIC SYSTEMS

Received February 21, 2002 DOI: 10.1070/RD2002v007n02ABEH000206

In this survey the basic concepts of nonholonomic systems such as equations of motion, the theory of reducing multiplier, variational principles and the Hamilton – Jacobi theorem are presented with some interesting bibliographical details.

1. Introduction

The term "nonholonomic system" was introduced in mechanics by H. Hertz [64]. It means that a material system is subjected to such kind of constraints that restrict the velocities of particles composed the system, but not their position (configuration of the system).

More precise definition is the following. Suppose that m vector fields $X_1, \ldots, X_m$ are defined in a domain $G(x_1, x_2, \ldots, x_n)$. The coordinates of the vectors X_i are $\xi_i^1(x_1, \ldots, x_n), \ldots, \xi_i^n(x_1, \ldots, x_n)$ and suppose that they are k times continuously differentiable functions. The vector X_i corresponds to the operator

$$X_i = \xi_i^1 \frac{\partial}{\partial x_1} + \ldots + \xi_i^n \frac{\partial}{\partial x_n}.$$

Trajectories of such operator X_i are specified by the differential equations

$$\frac{dx_1}{dt_i} = \xi_i^1(x_1, \ldots, x_n), \ldots \frac{dx_n}{dt_i} = \xi_i^n(x_1, \ldots, x_n),$$

where t_i is a parameter of trajectories of operator X_i.

At every point of n-dimensional domain G vectors X_i generate an m-dimensional direction. The field of these m dimensional directions is called completely nonholonomic if the system of partial differential equations

$$X_1 f = X_2 f = \ldots = X_m f = 0,$$

where $f(x_1, \ldots, x_n)$ is a desired function, has only the trivial solution $f \equiv \text{const}$. The sufficient condition for that is the existence of n operators linearly independent at all points of the domain G among the operators $X_1, \ldots, X_m$ and Poisson brackets of all possible their pairs (with calculating the next step-by-step Poisson brackets if necessary).

Suppose the operators $X_1, \ldots, X_m$ generate completely nonholonomic field of m-dimensional directions in the domain G. Then one can move from any point of the domain G to any other point of this domain by finite number of transitions along trajectories of operators $X_1, X_2, \ldots, X_m$ [24].

Nonholonomic constraints appear in systems with rigid bodies which roll along surfaces without sliding. As an example we consider a heavy homogeneous hoop of radius 1 that rolls and rotates on a fixed horizontal plane. The fixed OX and OY-axes are specified in the support plane, and the fixed OZ-axis is directed upward vertically. Let ξ, η, z be the coordinates of the hoop's mass center G

Mathematics Subject Classification 37J60 with respect to these axes and φ, ψ, θ be the Euler angles of rotational displacement of the hoop with respect to the König $OX'Y'Z'$-axes that are parallel to the fixed $OXYZ$- axes.

The equations of constraints are [48]:

$$\begin{aligned}
&\dot{\xi} - \dot{\theta}\sin\psi\sin\theta + \dot{\psi}\cos\psi\cos\theta + \dot{\varphi}\cos\psi = 0,\\
&\dot{\eta} + \dot{\theta}\cos\psi\sin\theta + \dot{\psi}\sin\psi\cos\theta + \dot{\varphi}\sin\psi = 0,\\
&\dot{z} - \dot{\theta}\cos\theta = 0.
\end{aligned} \tag{1.1}$$

The third equation express the finite geometric constraint $z = \cos\theta$, and the first and second equations present the nonholonomic constraints. Indeed, at each point $(\xi, \eta, \theta, \psi, \varphi)$ of the configurational manyfold they define the 3-dimensional direction having, as an example, basis vectors to

$$\begin{aligned}
X_1 &= \sin\psi\sin\theta\frac{\partial}{\partial\xi} - \cos\psi\sin\theta\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\theta},\\
X_2 &= -\cos\psi\cos\theta\frac{\partial}{\partial\xi} - \sin\psi\cos\theta\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\psi},\\
X_3 &= -\cos\psi\frac{\partial}{\partial\xi} - \sin\psi\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\varphi}.
\end{aligned}$$

The field of these 3-dimensional directions is completely nonholonomic since the operators X_1, X_2, X_3, and

$$\begin{aligned}
X_4 &= (X_2, X_3) = \sin\varphi\frac{\partial}{\partial\xi} - \cos\psi\frac{\partial}{\partial\eta},\\
X_5 &= (X_2, X_4) = \cos\psi\frac{\partial}{\partial\xi} - \sin\psi\frac{\partial}{\partial\eta}
\end{aligned}$$

are linearly independent at each point $(\xi, \eta, \theta, \psi, \varphi)$ of the configurational manifold. Hence the hoop can be moved from any position to another (in accordance with equations of constraints (1.1)) providing that finally the hoop touches an arbitrary point of the supporting plane with any given point of its circle has an arbitrarily defined orientation in the Euclidean space. However, this assertion is geometrically evident.

2. Examples of nonholonomic constraints ideal by Lagrange

The most broad class of ideal nonholonomic constraints occurring in classical mechanics is presented by systems having rigid bodies that roll or rotate without sliding along fixed or movable support surfaces provided that each body is supported by the surface at exactly one point. If $\boldsymbol{v} = 0$ is the instantaneous relative velocity of the point of a body which is the contact point with the support surface in the considered instant then for every possible (virtual) motion of the body the following condition is fulfilled $\delta\boldsymbol{v} = 0$ [48]. Therefore, the point force of reaction of the support $\boldsymbol{R}$ does not produce the virtual work for this motion: $R\cdot\delta r = 0$. The constraints implemented by forces that do not perform any work on *all* possible motions of system are called ideal in the Lagrange sense.

In addition to the presented class of ideal nonholonomic constraints there are some examples of mechanical systems with nonholonomic constraints of another nature which are also ideal by Lagrange.

The first example was presented by S. A. Chaplygin [40]. A rigid body touches the horizontal support plane in three points; two of them are freely sliding legs; the third one is the point of contact A of sharp wheel (blade) with horizontal axis fixed in the moving body. The wheel can not slide in the direction perpendicular to its plane. This kinematic condition is written in the following form:

$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0, \tag{2.1}$$

where (x, y) are the plane coordinates of A, φ is the angle between the fixed axis OX and the plane of the wheel. Physically, the sharp wheel as if cut into the support plane producing a shallow furrow, along which the wheel can easily slide, but that furrow prevents the wheel motion in the transversal direction. On the other hand, a rotation about the point A is possible motion (according to (2.1), but "furrow" also prevents this motion. To remedy this, the blade is supposed to be a wheel not a length (skate). The described system is sometimes referred to as "Chaplygin sledge".

The second example was given by G. K. Suslov in [34]. Two rigid bodies rotate about their fixed points and are connected with each other by long untwisted rope, which width is neglected. We suppose that the fixation of rope to the surface of body is such that the tangent $\boldsymbol{\tau}$ to the rope in the point of fixation is fixed with respect to the body. The untwisting condition implies the kinematic condition: the projection of instantaneous angular velocity of body's rotation on the direction $\boldsymbol{\tau}$ is zero. This is the nonholonomic constraint. One endpoint of the rope can be connected to the axis of clockwork ($\omega_{\boldsymbol{\tau}} = \text{const}$) or fixed ($\omega_{\boldsymbol{\tau}} = 0$).

P. V. Woronetz [5] considered general linear integrals of some holonomic systems with zero values of integration constants as nonholonomic constraints. Strictly speaking, the first integrals do not put any kinematical restrictions on the system, since their realization does not require applying additional forces. But formally the condition $R \cdot \delta r = 0 \cdot \delta r = 0$ is satisfied. Therefore we can reduce the order of motion equations of such holonomic system using the equations forms developed for nonholonomic systems.

All the previous examples concern the case of ideal nonholonomic constraints linear with respect to velocities. Theoretically we may consider nonlinear nonholonomic constraints, but constructing them and verification of their property not produce any virtual work are still problematic. In [49] the system with ideal linear nonholonomic constraints was proposed. Then, using the limit passage to zero value of one of the geometric parameters of this system and, as a result, reducing the order of differential equations of motion (see also [62, 18]), the system with the nonlinear constraint

$$\dot{x}^2 + \dot{y}^2 = a^2 \dot{z}^2 \qquad (a = \text{const}) \tag{2.2}$$

and Lagrangian $L = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2 - mgZ$ of heavy particle with mass m was obtained. This example by Appell is practically the only one that illustrates some general theoretical constructions in mechanics of systems with ideal by Lagrange nonlinear nonholonomic constraints.

3. Equations of motion

J. L. Lagrange has proposed the universal form of equations of motion for system with ideal constraints. Moreover the equivalence of the constraints equations written in the differential form

$$\sum_{j=1}^{n} a_{ij}(t, q)\delta q_i = 0 \qquad (i = k+1, \ldots, n) \tag{3.1}$$

to the finite form

$$f_i(t, q) = c_i \tag{3.2}$$

does not required. They are the Lagrange equations of the first kind with the multipliers λ_i of constraints:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j + \sum \lambda_i a_{ij} \tag{3.3}$$

(in generally accepted notations).

If a mechanical system is subjected to finite ideal constraints (3.2) only (rank $\left\|\dfrac{\partial f}{\partial q}\right\| = n-k$) then the order of equations (3.3) can be reduced by the number $2(n-k)$ [67]:

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot q_s} - \frac{\partial T^*}{\partial q_s} = Q_s^* \qquad (s=1,\,\ldots,\,k). \tag{3.4}$$

Here $T^*(t, q_1,\,\ldots,\,q_k, \dot q_1,\,\ldots,\,\dot q_k)$ is a contraction of function $T(t, q_1,\,\ldots,\,q_n, \dot q_1,\,\ldots,\,\dot q_n)$ on the tangent foliation of configurational manifold (3.2) of system; the generalized forces

$$Q_s^* = Q_s + \sum Q_i b_{is}$$

correspond to locally independent generalized (lagrangian) coordinates $q_1,\,\ldots,\,q_s$, and equations (3.1) are rewritten in the form

$$\delta q_i = \sum_{s=1}^{k} b_{is}\delta q_s \qquad (i=k+1,\,\ldots,\,n). \tag{3.5}$$

In the general case the equations of motion for systems with nonholonomic constraints do not have form (3.4) of the Lagrange equations of the second kind. Apparently, N. Ferrers [60] was the first who has pointed out this fact. Nevertheless, when decades passed after his work, several publications concerning solutions of some problems with nonintegrable constraints have appeared with erroneous application of Lagrange equations (3.4) [54, 69], the Hamilton principle [71], and the Jacobi method [57].

Following Ferrers, we put the equations of nonholonomic constraints in the form

$$\dot x = \Theta\cdot\dot\theta + \Phi\cdot\dot\varphi + \Psi\cdot\dot\psi + \ldots, \tag{3.6}$$

where $\dot x = (\dot x_1, \dot x_2,\,\ldots)$ are "dependent velocities" of particles of system, $(\theta, \varphi, \psi,\,\ldots)$ are "generalized coordinates that express the vive force $2T$ and potential energy U". Since we have

$$\frac{d}{dt}\left(\frac{1}{2}\frac{\partial \dot x^2}{\partial \dot\theta}\right) = \dot x\frac{d\Theta}{d\theta} + \ddot x\Theta \quad \text{etc.},$$

the d'Alembert principle

$$\sum(m\ddot x - F_x)\delta x = \sum m\ddot x\delta x - \delta U = \sum\left[\left(m\ddot x\Theta - \frac{\partial U}{\partial\theta}\right)\delta\theta + \ldots\right] = 0$$

written for the considered system of particles gives

$$\frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \sum m\dot x\frac{d\Theta}{dt} = \frac{\partial U}{\partial\theta}, \tag{3.7}$$

and etc. for each generalized coordinate.

When equations of constraints (3.6) are integrable then

$$\frac{\partial\Theta}{\partial\varphi} = \frac{\partial\Phi}{\partial\theta},\,\ldots,\,,$$

hence

$$\frac{d\Theta}{dt} = \frac{\partial\dot x}{\partial\theta},\,\ldots,$$

and equations (3.7) become the Lagrange equations of the second kind. In particular, if for any generalized coordinate, for example, for θ, the conditions

$$\frac{\partial\Theta}{\partial\varphi} = \frac{\partial\Phi}{\partial\theta}, \qquad \frac{\partial\Theta}{\partial\psi} = \frac{\partial\Psi}{\partial\theta},\,\ldots, \tag{3.8}$$

are fulfilled then equation (3.7) for θ become the Lagrange equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} = \frac{\partial U}{\partial \theta}.$$

Ferrers has found that conditions (3.8) are fulfilled for the angle of nutation θ in the problem of rolling of a heavy homogeneous hoop on a horizontal plane (see equations of constraints (1.1)).

25 years later, S. A. Chaplygin in [41] studied the class of nonholonomic systems that was considered by Ferrers. Now these systems are called the Chaplygin systems [18]. Suppose the generalized coordinates of the system are divided into two sets $(q_1, \ldots, q_k)$ and $(q_{k+1}, \ldots, q_n)$ so that the kinetic energy T, force function U, and equations of constraints

$$\dot q_i = \sum_{s=1}^{k} b_{is}(q)\dot q_s \qquad (i = k+1, \ldots, n) \tag{3.9}$$

of the system do not depend on coordinates $(q_{k+1}, \ldots, q_n)$. Chaplygin also takes into account the d'Alembert–Lagrange principle and, using equations of constraints (3.9), excludes the dependent velocities $\dot q_i$ from the expression of kinetic energy T. As a result he obtains the known Chaplygin equations:

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot q_s} - \frac{\partial T^*}{\partial q_s} - \frac{\partial U}{\partial q_s} = \sum_{i=k+1}^{n}\left(\frac{\partial T}{\partial \dot q_i}\right)^{*}\left\{\sum_{r=1}^{k}\left(\frac{\partial b_{is}}{\partial q_r} - \frac{\partial b_{ir}}{\partial q_s}\right)\dot q_r\right\}, \tag{3.10}$$

that clearly differ from the Lagrange equations of the second kind (3.4) $\left(Q_s^* = \dfrac{\partial U}{\partial q_s}\right)$ because the additional "nonholonomic terms" are presented.

Curiously, Chaplygin never used equations (3.10) to examine specific nonholonomic systems. He preferred the general theorems of dynamics to obtain the equations of motion and the first integrals. Such approach was typical for some other classics of science [74, 11, 68].

P. V. Woronetz [6] deduced the equations of motion of conservative systems with linear nonholonomic constraints in the general case, when T, U and b_{is} explicitly depend on all generalized coordinates:

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot q_s} - \sum_{i=k+1}^{n}\frac{\partial (T^*+U)}{\partial q_i}b_{is} - \frac{\partial (T^*+U)}{\partial q_s} =$$
$$= \sum_{i=k+1}^{n}\left(\frac{\partial T}{\partial \dot q_i}\right)^{*}\left\{\sum_{r=1}^{k}\left(\frac{\partial b_{is}}{\partial q_r} + \sum_{p=k+1}^{n}\frac{\partial b_{is}}{\partial q_p}b_{pr} - \frac{\partial b_{ir}}{\partial q_s} - \sum_{p=k+1}^{n}\frac{\partial b_{ir}}{\partial q_p}b_{ps}\right)\dot q_r\right\} \qquad (s = 1, \ldots, k).$$

Contrary to (3.10), the Woronetz equations can not be separated from equations of nonholonomic constraints (3.9) and must be considered jointly. Woronetz has applied his equations to the classical problem of body's rolling on an arbitrary surface [7].

P. Appell in the first edition of "Traité de Mécanique Rationelle" has introduced the Lagrange equations of the second kind in the chapter concerned with the nonholonomic systems. The error was corrected in the next editions of the treatise. At the same time the author has devoted a number of papers to the systems with nonintegrable constraints. As a result, there was obtained [50] the following form of equations of motion of nonholonomic systems

$$\frac{\partial S}{\partial \ddot q_s} = Q_s \qquad (s = 1, \ldots, k < n), \tag{3.11}$$

where $S = \frac{1}{2}\sum m(\ddot x^2 + \ddot y^2 + \ddot z^2)$ is the energy of accelerations calculated if we use the equations of constraints.

Equations (3.11) for systems with holonomic bilateral and unilateral constraints in the coordinate and quasicoordinate form were used even earlier by D. Gibbs [61].

In one of his papers [54] L. Boltzmann considered a mechanical system with a friction gear: driving and driven wheels are immovable on the mutually perpendicular shafts, there is no sliding at the point of contact of the wheels, and the distance from this point up to the center of the driving wheel is a regulated function of time. Thus, the angular velocities of wheels satisfy the nonholonomic relation

$$\omega = a(t)w.$$

But in spite of the fact that the coordinate ω is nonholonomic (quasicoordinate), Boltzmann wrote out the Lagrange equations of the second kind. After 17 years [55] he returned to this system, corrected the error, and deduced the general equations of motion of nonholonomic systems in quasicoordinates. These equations were somewhat generalized by G. Hamel [63] and they are now referred to as the Boltzmann–Hamel equations. For stationary systems these equations have form

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_s} - \frac{\partial T}{\partial \pi_s} + \sum_{j=1}^{n}\sum_{r=1}^{k}\frac{\partial T}{\partial \dot{\pi}_j}\gamma^j_{rs}\dot{\pi}_r\right)\Bigg|_{\dot{\pi}_{k+1}=\dot{\pi}_{k+2}=\ldots=\dot{\pi}_n=0} = \Pi_s \qquad (s = 1, \ldots, k). \tag{3.12}$$

In these equations the kinetic energy of the system is expressed in quasi-velocities $\dot{\pi}_1, \ldots, \dot{\pi}_n$ (kinematic characteristics)

$$T(q, \dot{q}) = T(q, \dot{\pi}).$$

The latter are specified trough the generalized velocities $\dot{q}$ by the linear equalities

$$\dot{\pi}_j = \sum_{h=1}^{n} a_{jh}(q)\dot{q}_h \qquad (\det ||a_{jh}|| \neq 0),$$

and the right-hand sides of the last $n-k$ equalities represent the equations

$$\sum_{i=k+1}^{n} a_{ih}\dot{q}_h = 0 \qquad (i = k+1, \ldots, n)$$

of constraints imposed on the system. Three-index symbols are

$$\gamma^j_{hp} = \sum_{u=1}^{u}\sum_{v=1}^{n}\left(\frac{\partial a_{jv}}{\partial q_u} - \frac{\partial a_{ju}}{\partial q_v}\right)c_{uh}c_{vp}, \qquad ||c_{uv}|| = ||a_{jh}||^{-1} \qquad (i, h, p = 1, \ldots, n).$$

The right-hand sides of (3.12) are the generalized forces corresponding to independent variations of quasicoordinates $\delta\pi_1, \ldots, \delta\pi_k$.

The rules of calculating of three-index symbols, and also the generalization of equations (3.12) for the case of non-stationary systems with constraints, and the examples of deductions of these equations are presented in [17]. Following Hamel, A. I. Lur'e called them the Euler–Lagrange equations. They represent the most general equations of dynamics.

The analogous equations in kinematic characteristics have been obtained by V. Volterra [77].

Note also the so-called Poincare–Chetaev equations [37, 38, 39, 25, 26, 27]. Developing optical-mechanical analogy, W. Hamilton has discovered that the process of motion of a holonomic conservative system can be considered as the permanent development of the contact transformation, and the set of contact transformations has the group property [78]. This representation of motion has been advanced by H. Poincaré [73]. For conservative holonomic systems with k degrees of freedoms he proposed the new form of the equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_s} = \sum_{l=1}^{k}\sum_{r=1}^{k} c_{lsr}\frac{\partial T}{\partial \eta_r}\eta_l + X_s(T+U) \qquad (s = 1, \ldots, k), \tag{3.13}$$

in which the kinematic energy T of a system is not expressed in the generalized coordinates and velocities $(q, \dot{q})$ as in the Lagrange equations, but in the variables (q, η), where $\eta_s dt$ $(s = 1, \ldots, k)$ are parameters of actual transitions. Transition from a state $(q_1, \ldots, q_k)$ of the system to the infinitesimally close state $(q_1 + \dot{q}_1 dt, \ldots, q_k + \dot{q}_k dt)$ is produced by the infinitesimal transformation of the group

$$\sum_{s=1}^{k} \eta_s dt\, X_s(f),$$

so that

$$\dot{q}_r = \frac{dq_r}{dt} = \sum_{s=1}^{k} \eta_s X_s^r \qquad (r = 1, \ldots, k).$$

Here $\left\{ X_l = \sum\limits_{s=1}^{k} X_l^s(q) \dfrac{\partial}{\partial q_s} \right\}$ is the closed set of infinitesimal operators, which defines the continuous group according to the second inverse theorem of S. Lie. For the closed set of operators the commutator is

$$X_r X_l - X_l X_r = \sum_{s=1}^{k} c_{rls} X_s,$$

where c_{rls} are the structural *constants* of the group.

Equations (3.13) were generalized by N. G. Chetaev for the case of holonomic systems defined in redundant generalized coordinates $q_1, \ldots, q_k, q_{k+1}, \ldots, q_n$ which are related by $n-k$ finite equations. The corresponding group of transformations is intransitive. Chetaev [43, 44] has constructed the complete theory of such equations and considered examples.

Fam Guen generalized the technique (and terminology) of deduction of the motion equations of mechanical systems in variables (q, η) on nonholonomic systems with linear constraints. But generally a motion of nonholonomic systems has no group property as in holonomic systems, three-index symbols $c_{\alpha\beta\gamma}$ for the closed set of differential operators are not constants, and the quantities $\eta_s dt$ are essentially the quasicoordinates. Therefore, equations in the Poincaré–Chetaev variables are identical to the Volterra and Boltzmann–Hamel equations in quasicoordinates [8, 38, 13].

When deriving the equations of motion of a system with ideal constraints, which are linear relative to the velocities, it does not matter what fundamental principle of mechanics is taken as a base. It could be the Gauss principle of least compulsion (Zwung) or the d'Alembert–Lagrange principle. These both differential principles are equivalent in the case of linear ideal constraints [70, 4]. If the following nonlinear constraints are imposed on a mechanical system

$$f_i(t, q, \dot{q}) = 0 \qquad \left(\operatorname{rank} \left\| \frac{\partial f_i}{\partial \dot{q}_i} \right\| = n - k\right) \qquad (i = k+1, \ldots, n), \tag{3.14}$$

then it is necessary to define the concept of virtual displacements so that both principles imply the same equations of motion, and this definition of virtual displacements in the case of linear constraints agrees with the known one.

This problem was solved by N. G. Chetaev [45]. Moreover V. I. Kirgetov proved the following statement.

Suppose the ideal constraints of the second order are imposed on a system

$$a_{i1}\ddot{q}_1 + \ldots + a_{in}\ddot{q}_n = a_i \qquad (i = k+1, \ldots, n)$$

$(a_{ij}, a_i$ are functions of $t, q, \dot{q})$. These relations are the total derivatives of (3.14) with respect to time along the trajectories satisfying equations of constraints (3.14), therefore

$$a_{ij} = \frac{\partial f_i}{\partial \dot{q}_j}, \qquad a_i = -\frac{\partial f_i}{\partial t} - \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\dot{q}_j \qquad (i = k+1, \ldots, n; \quad j = 1, \ldots, n).$$

We suppose that constraints (functions a_{ij}, a_i) do not depend on the forces imposed on the system. It turns out [14] that if virtual displacements do not depend on forces acting on the system and the d'Alembert– Lagrange principle implies the same equations as the Gauss principle then among all possible linear definitions of virtual displacements there is no such one that is not equivalent to the Chetaev definition:

$$\frac{\partial f_i}{\partial \dot{q}_1}\delta q_1 + \ldots + \frac{\partial f_i}{\partial \dot{q}_j}\delta q_j = 0 \qquad (i = k+1, \ldots, n). \tag{3.15}$$

For systems with nonlinear constraints which reactions do not produce the virtual work on any virtual displacement defined by relations (3.15), the equation of motion are written in form (3.3). However, for such systems we can derive the equations of motion without multipliers. For example, for the nonlinear Chaplygin systems with constraints

$$\dot{q}_i = e_i(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k) \qquad (i = k+1, \ldots, n) \tag{3.16}$$

(T and U do not depend on $q_{k+1}, \ldots, q_n$) the equations of motion have form [19]

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_s} - \frac{\partial T^*}{\partial q_s} - \frac{\partial U}{\partial q_s} = \sum_{i=k+1}^{n}\left(\frac{\partial T}{\partial \dot{q}_i}\right)^*\left(\frac{d}{dt}\frac{\partial e_i}{\partial \dot{q}_s} - \frac{\partial e_i}{\partial q_s}\right) \qquad (s = 1, \ldots, k) \tag{3.17}$$

(The asterisk near some expressions marks the result of the replacement of the quantities $\dot{q}_{k+1}, \ldots, \dot{q}_n$ with the right-hand sides of equations (3.16) in the corresponding expressions). Note that in the case of nonlinear constraints (3.16) the function T^* can be degenerate with respect to $\dot{q}_i, \ldots, \dot{q}_k$.

So, the various kinds of the motion equations of nonholonomic systems do not have the Lagrangian form because of nonintegrability of the constraints' equations. And still there are some examples of mechanical systems, which general motion is exactly described by the Lagrange equations of second kind or by the Hamilton equations with the degenerate Hamiltonian.

First, a heavy homogeneous hoop (§ 1) rolling on a horizontal plane gives such an example provided that the angle $\theta \equiv \frac{\pi}{2}$ in time of motion (this additional geometrical constraint may be easily realized). The two first equations of constraints (1.1) become

$$\dot{\xi} + \dot{\varphi}\cos\psi = 0, \qquad \dot{\eta} + \dot{\varphi}\sin\psi = 0$$

and still remain non-integrable. Thus the right-hand sides of Chaplygin equations (3.10) written in the generalized coordinates φ, ψ vanish identically.

The other example is described in J. Synge's book [76], and the fact that the equations of motion in this example have the Lagrange form, apparently, for the first time was noted in [46]. In the Synge example two identical homogeneous wheels are freely mounted on a common axis and rolled on a fixed horizontal plane without sliding. The equations of motion of this nonholonomic system also have the form of the Lagrange equations of the second kind with respect to the Lagrange function $L^* = T^*$.

The third example is the Appell system (§ 2). If equation of constraint (2.2) is written as

$$\dot{z} = \pm\frac{1}{a}\sqrt{\dot{x}^2 + \dot{y}^2},$$

then the system is not a Chaplygin one, however the equations of motion of the system can be presented as the Hamilton equations, although with the degenerate Hamiltonian [75].

As the fourth example we consider a problem of rolling and rotation of a homogeneous ball on a horizontal plane. This problem illustrates the Woronetz idea to interpret the first integrals of a system as nonholonomic constraints (§ 2). Indeed, the projections of the instantaneous velocity of the contact point of the ball with the supporting plane onto the fixed orthogonal axes of this plane are

$$v_x = \dot{x} - qR, \qquad v_y = \dot{y} + pR,$$

where R is the radius, $(\dot{x}, \dot{y})$ are the projections of the velocity of the ball's center, (p, q, r) are projections of the angular velocity onto any three orthogonal axes fixed in the ball. It is clear that if the ball slides along a frictionless horizontal plane (the holonomic system) then $v_x = \text{const}$ and $v_y = \text{const}$ are the first integrals of the equations of motion. Hence, if initial conditions of motion of a holonomic system are chosen so that $v_x = v_y = 0$ then during the ball motion the conditions

$$\dot{x} - qR = 0, \qquad \dot{y} + pR = 0$$

are fulfilled automatically. Hence, the equations of homogeneous ball's motion on a horizontal plane without sliding have the form of the Lagrange equations of the second kind with function $L = T$ (not T^*!) since the multipliers of constraints in equations (3.3) are identically equal to zero.

In spite of the enumerated exceptions, in the general case the equations of motion of nonholonomic systems do not have the Lagrangian form, therefore many theorems of an analytical mechanics proved for systems with finite constraints can not be generalized for nonholonomic systems. This is why the possibility of transformation of the motion equations of nonholonomic systems to the form of the Lagrange or Hamilton equations is the problem of examining in a number of papers.

In XIX century the popular method of research of analogy between different mechanical systems was direct constructing the transformations of the motion equations of one system into the equations of the other. Such papers were published by Liouville, Goursat, Painleve, Stekkel, Bertrand, Dáuteville, Darboux, and Appell. For example, Appell in [51] proposed to use a point transformation of coordinates with the transformation of the independent variable

$$dt = \lambda(q_1, \, \ldots, \, q_n) \, dt_1$$

and to specify the function λ so that in the new equations of motion of the Lagrange form the forces depending on coordinates and velocities ("nonholonomic terms" in equations (3.10) can be interpreted in such a way) become additional generalized *positional* forces. Appell has put this problem of elimination of right-hand sides in equations (3.10) completely clear in 1901 [52]. Some years later Chaplygin realized Appell's idea in [40] developing the theory of reducing multiplier for nonholonomic systems with two degrees of freedom.

According to Chaplygin, in nonholonomic systems with two degrees of freedom (Chaplygin systems) a new independent variable can be always introduced

$$d\tau = N(q_1, q_2) \, dt$$

so that in the variables q_1, q_2, τ the equations of motion have the Lagrange and Hamilton forms, and "this technique is interesting from the theoretical point of view as a direct generalization of Jacobi's methods for the elementary cases of a nonholonomic system" [40].

There are two cases in the problem of determination of the reducing multiplier $N(q_1, q_2)$. In the first case the function N is obtained by a quadrature, and the coefficients of the equations of constraints and the kinetic energy must identically satisfy the unique condition of compatibility. In the second case, when the condition of compatibility is not fulfilled, the problem of integration of the canonical motion equations of the system is reduced to the sequential integration of two first order partial differential equations.

Chaplygin illustrated his theory by four examples, three of them were solved by quadratures [40, 42]. One more example was given by E. I. Kherlamova [12]. In all these examples the reducing multiplier is determined by one quadrature.

In the example of "sledge" (§ 2) Chaplygin rewrote equation of constraint (2.1) as

$$\dot{x} = \dot{s} \cos \varphi, \qquad \dot{y} = \dot{s} \sin \varphi,$$

introducing redundant nonholonomic coordinate $s \, (ds = \sqrt{dx^2 + dy^2}\,)$. And, though the theory of reducing multiplier was constructed for the case of the curvilinear coordinates, the general law of motion of the sledge found by Chaplygin was confirmed by M. I. Efimov [10], who integrated this problem in coordinates (x, y, φ). It turns out that in these coordinates the reducing multiplier is obtained by quadrature only if the center of mass of sledge is orthogonally projected on the rectilinear track of the cutting wheel. If the center of mass is displaced in any side then the problem must be solved by the sequential integration of two first order partial differential equations (one of them is nonlinear). It seems that this example is the only one up to now that illustrates the second case of Chaplygin's theory.

In addition to the Chaplygin solution of the problem of "sledge" motion we note another result presented by I. S. Arzhanykh [1]. Suppose that the equations of linear constraints have the form

$$\dot{q}_i = b_{i1}\dot{q}_1 + b_{i2}\dot{q}_2 \qquad (i = 3, \ldots, n)$$

and the coefficients of kinetic energy of the Chaplygin nonholonomic system depend only on one coordinate, for example, on q_1. We introduce the additional variable η:

$$\dot{q}_i = A_i\dot{\eta} + B_i\dot{q}_1, \qquad \dot{q}_2 = A\dot{\eta} + B\dot{q}_1 \qquad (i = 3, \ldots, n), \tag{3.18}$$

where $A_i = b_{i2}A$, $B_i = b_{i1} + b_{i2}B$. It turns out that we can always choose functions $A(q_1)$, $B(q_1)$ so that the reducing multiplier of the system with constraints (3.18) is determined by one quadrature (i. e. the first case of Chaplygin theory always takes place).

Attempts to generalize the theory of reducing multiplier on nonholonomic systems having the number of degrees of freedom more than two are ineffective. For example, for the Chaplygin systems with three degrees of freedom the case similar to the first one in the Chaplygin theory takes place if the coefficients of the kinetic energy and of equations of constraints identically satisfy six independent identities. And in the case similar to the second one in the Chaplygin theory we have to integrate two partial differential equations, however, they are of the second order [10].

Contrary to the method of reducing multiplier, we can try to transform equation (3.10) to the Lagrange form without change of the independent variable. Such statement leads to the inverse problem of calculus of variations. Its particular case is the following problem:

Let the system of differential equations

$$F_s(t; q_1, \ldots, q_k; \dot{q}_1, \ldots, \dot{q}_k; \ddot{q}_1, \ldots, \ddot{q}_k) = 0, \qquad \det\left\|\frac{\partial F_s}{\partial \ddot{q}_r}\right\| \neq 0 \qquad (s = 1, \ldots, k). \tag{3.19}$$

be given. Determine what the necessary and sufficient conditions must be satisfied by functions F_i to make system (3.19) equivalent (in sense of identity of their general solutions) to the following differential system

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_s} - \frac{\partial \Theta}{\partial q_s} = 0 \qquad (s = 1, \ldots, k),$$
$$\Theta = \Theta(t, q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k), \qquad \det\left\|\frac{\partial^2 \Theta}{\partial \dot{q}_s \partial \dot{q}_r}\right\| \neq 0. \tag{3.20}$$

The function Θ is referred to as the kinetic potential.

In such statement the problem has constructive solutions for $k = 1$ [31, 58] and $k = 2$ [59]. There are no essential applications of its solution for nonholonomic systems [23, 2, 3, 19, 20, 15, 16]. In the general case Chaplygin equations (3.10) can not be represented as (3.20) [32].

4. Generalizations of the Hamilton principle

Let's consider any holonomic system with the independent coordinates $q_1, \ldots, q_n$ and Lagrange function $L(t, q, \dot{q})$. If functions $q_j = q_j(t)$ $(j = 1, \ldots, n)$ are specified arbitrarily then we obtain the motion possible according to the constraints implied on the system. Let $\{q(t, \alpha)\}$ be an one-parameter family of all possible motions transferring the system from the given initial position $\{q^0\}$ at the instant t_0 to the given final position $\{q^1\}$ at the instant t_1. The initial and final instants t_0 and t_1, and the initial and final position of the system are fixed in advance. In all other respects the motions are arbitrary. The parameter varies in the range $(-\varepsilon \leqslant \alpha \leqslant \varepsilon)$; the motion at $\alpha = 0$ is true (real) motion of the holonomic system, and the motions at $\alpha \neq 0$ are adjoining motions. So, we have

$$q(t_0, \alpha) = q^0, \qquad q(t_1, \alpha) = q^1 \qquad (-\varepsilon \leqslant \alpha \leqslant \varepsilon).$$

The Hamilton principle states that on the real motion the functional of action

$$W(\alpha) = \int\limits_{t_0}^{t_1} L[t, q(t, \alpha), \dot{q}(t, \alpha)]\, dt$$

has the stationary value, i. e.

$$\delta \int\limits_{t_0}^{t_1} L(t, q, \dot{q})\, dt = 0, \tag{4.1}$$

where

$$\delta q_j = \left.\frac{\partial q_j}{\partial \alpha}\right|_{\alpha=0} \delta\alpha \qquad (j = 1, \ldots, n)$$

are isochronous variations of coordinates, and [9]

$$\frac{d}{dt}\delta q_j = \delta \frac{dq_j}{dt} \qquad (t_0 < t < t_1), \qquad \delta q_j\Big|_{t=t_0} = \delta q_j\Big|_{t=t_1} = 0 \qquad (j = 1, \ldots, n). \tag{4.2}$$

Now let's consider an arbitrary nonholonomic system with a Lagrange function $L(t, q, \dot{q})$ and the ideal nonintegrable linear constraints

$$a_{i_1}(t, q)\dot{q}_1 + \ldots + a_{in}(t, q)\dot{q}_n + a_i(t, q) = 0, \qquad \operatorname{rank} ||a_{ij}|| = n - k \qquad (i = 1, \ldots, n-k). \tag{4.3}$$

The generalized coordinates $q_1, \ldots, q_n$ of the system are independent since they are not related by any finite relations (§ 1), and the generalized velocities $\dot{q}_1, \ldots, \dot{q}_n$ must satisfy nonintegrable equations (4.3) identically with respect to t.

Let $\{q^0\}$ be the given initial position of the system at the instant t_0. In the case of holonomic system the point $\{q^1\}$ that is the final position of the system at the time t_1 is chosen more or less arbitrarily on the configuration manifold, but for the nonholonomic system the point $\{q^1\}$ can be chosen only from the set of positions which are passed from the initial position $\{q^0\}$ [72]. In other words we have to know in advance that the points $\{q^0, t_0\}$ and $\{q^1, t_1\}$ belong to the specified real trajectory of the system.

Now there is an infinite set of adjoining motions of the system which are possible according to the linear *nonintegrable* constraints; these motions pass through the indicated points. If for isochronous variations of coordinates we assume conditions (4.2), but not relations

$$a_{i1}(t, q)\delta q + \ldots + a_{in}(t, q)\delta q_n = 0 \qquad (i = 1, \ldots, n-k), \tag{4.4}$$

that are satisfied with virtual displacement of a system then the Hamilton principle in form (4.1) is not fulfilled for this nonholonomic system.

Indeed, calculating the variation of the functional of action and using as generally permutational relations and boundary conditions (4.2), we obtain the nonzero expression

$$\int\limits_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j\, dt,$$

(the integrand is the negative value of the virtual work of constraints' reaction on a virtual displacement that, generally speaking, does not satisfy (4.4)).

So, to generalize principle (4.1) for nonholonomic systems we must take into account that

(a) all adjoining motions closed to the specified real trajectory are possible, i. e.

$$\delta\omega_1 = \ldots = \delta\omega_{n-k} = 0$$

(ω_i denote the left-hand sides of equations (4.3));

(b) isochronous variations $\delta q_1, \ldots, \delta n$ are the virtual displacement (4.4) of the system under consideration;

(c) permutational relations are fulfilled

$$\frac{d}{dt}\delta q_j = \delta \frac{dq_j}{dt} \qquad (j = 1, \ldots, n).$$

However, the enumerated three groups of conditions *are incompatible* in the case of nonintegrable constraints (4.3) [53, 18]. Hence, some conditions from (a), (b), (c) must be excluded.

If we exclude condition (a), but keep (b) and (c) then the set of adjoining motions passing through the given points $\{q^0\}$ and $\{q^1\}$ in instants t_0 and t_1 accordingly is defined by the formula

$$q_j(t) = \widetilde{q}_j(t) + \delta q_j(t) \qquad (j = 1, \ldots, n), \tag{4.5}$$

where $\{\widetilde{q}(t)\}$ is a real trajectory that includes points $\widetilde{q}(t_0) = q^0$ and $\widetilde{q}(t_1) = q^1$, and $\{\delta q\}$ are infinitesimal virtual displacements of the system defined by equalities (4.4). In the class of compared motions (4.5) the functional of action has stationary value for the real trajectory. This fact may be written in the form

$$\int\limits_{t_0}^{t_1} \delta L\, dt. \tag{4.6}$$

This is the Hamilton principle for nonholonomic systems in the Hölder form [65].

Note the principal difference between forms (4.1) and (4.6) of the Hamilton principle. For holonomic systems we know only two points $\{q^0, t_0\}$ and $\{q^1, t_1\}$ of the unknown real trajectory. But (4.1) implies the Euler–Lagrange equations which determine the desired motion for all values of time $t_0 \leqslant t \leqslant t_1$. On the contrary, (4.6) makes sense if the set of adjoining motions (4.5) is completely specified. But for this we have to know the whole segment of real trajectory $\{\widetilde{q}(t)\}$, $t_0 \leqslant t \leqslant t_1$ (see (4.5)) This is why the Hamilton principle in form (4.6) has more theoretical and less practical meaning.

We can keep conditions (a) and (b) but not require the fulfillment of permutational relations for *all* coordinates. Let's proceed in the following way. We solve equations of constraints (4.3) with respect to $n - k$ generalized velocities. Without loss of generality, one can write the result:

$$\dot{q}_i = b_{i1}(t, q)\dot{q}_1 + \ldots + b_{ik}(t, q)\dot{q}_k + b_i(t, q) \qquad i = (k + 1, \ldots, n).$$

Virtual displacements are

$$\delta q_i = b_{i1}\delta q_1 + \ldots + b_{ik}\delta q_k \qquad (i = k + 1, \ldots, n).$$

Assume the permutational relations are fulfilled for the first k coordinates but, basing on conditions (a) and (b), for the next $n - k$ coordinates from conditions (a) and (b) we derive that

$$\frac{d}{dt}\delta q_i - \delta\dot{q}_i = \frac{d}{dt}(b_{i1}\delta q_1 + \ldots + b_{ik}\delta q_k) - \delta(b_{i1}\dot{q}_1 + \ldots + b_{ik}\dot{q}_k + b_i) =$$

$$= \frac{db_{i1}}{dt}\delta q_1 + \ldots + \frac{db_{ik}}{dt}\delta q_k - \dot{q}_1\delta b_{i1} - \ldots - \dot{q}_k\delta b_{ik} - \delta b_i = \sum_{s=1}^{k}\sum_{r=1}^{k} A^i_{rs}\dot{q}_s\delta q_r - \delta b_i \qquad (i = k+1, \ldots, n).$$

With the help of these formulae d'Alembert–Lagrange principle is transformed to

$$\int\limits_{t_0}^{t_1} \left[\delta L + \sum_{i=k+1}^{n} \frac{\partial T}{\partial \dot{q}_i} \left(\sum_{s=1}^{k} \sum_{r=1}^{k} A^i_{rs} \dot{q}_s \delta q_r - \delta b_i \right) \right] dt = 0. \tag{4.7}$$

This is the Hamilton principle for nonholonomic systems in the Suslov form [34].

Now let's consider the case of nonholonomic system with constraints nonlinear relative to velocities (3.14). In nonholonomic systems with nonlinear constraints the set of possible motions passing through the given two points on the real trajectory can be *empty* [56, 66]. Such situation occurs in the Appell example (§ 2) when the particle moves by inertia ($g = 0$). In this case the real trajectory of the particle is a straight line. Let at initial instant $t = t_0$ the particle be in position $M_0(x_0, y_0, z_0)$ and at instant $t = t_1$ it be in position $M_1(x_1, y_1, z_1)$. Denote the orthogonal projections of these positions onto the coordinate plane $z = 0$ by $P(x_0, y_0)$ and $Q(x_1, y_1)$ accordingly. If another trajectory satisfies (2.2) and passes through M_0 and M_1 at the same instants as the real one then the integral of (2.2) along this curve with respect to time lead to contradiction immediately (assume $a = 1$ for simplicity)

$$z(t_1) = z(t_0) + \int\limits_{t_0}^{t_1} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt = z_0 + \int\limits_{P}^{Q} \sqrt{dx^2 + dy^2} > z_0 + \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = z_1.$$

Thus, in systems with nonlinear nonholonomic constraints the original conditions of the Hamilton principle may be unfulfilled.

However, the closer analysis of these requirements for holonomic systems shows that the conditions of fixation of adjoining motions at the endpoints can be replaced by the weaker conditions

$$\delta q_j \Big|_{t=t_0} = \frac{\partial q_j}{\partial q_\alpha} \Big|_{\substack{\alpha=0 \\ t=t_0}} \delta\alpha = 0, \qquad \delta q_j \Big|_{t=t_1} = \frac{\partial q_j}{\partial \alpha} \Big|_{\substack{\alpha=0 \\ t=t_1}} \delta\alpha = 0 \qquad (j = 1, \ldots, n). \tag{4.8}$$

In this case we can still obtain the Lagrange equations of motion by making variation (4.1) with the help of permutational relations and making the integration by parts under conditions (4.8).

In the case of nonholonomic constraint

$$\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

and, for example, of the real trajectory

$$x = t, \qquad y = 0, \qquad z = t \qquad (0 \leqslant t \leqslant 1),$$

we can choose as one-parameter family of adjoining motions

$$x = t, \qquad y = \alpha \sin 2\pi t, \qquad z = \int\limits_0^t \sqrt{1 + (2\pi\alpha \cos 2\pi t)^2}\, dt \qquad (0 \leqslant t \leqslant 1).$$

The real trajectory belongs to this family of curves. The origin of coordinates is the common point of all these curves but their endpoints are different while the variations

$$\delta x,\ \delta y = \delta\alpha \cdot \sin 2\pi t, \qquad \delta z = \delta\alpha \int\limits_0^t \frac{2\pi \cos 2\pi t}{\sqrt{1 + (2\pi\alpha \cos 2\pi t)^2}}\, dt$$

at $t = 1$ take zero values.

The following general conclusion is valid: in Chaplygin systems with regular constraints (3.16) such one-parameter family of kinematic possible curves adjoining to the segment M_0M_1 $(t_0 \leqslant t \leqslant t_1)$ of a real trajectory exists that conditions (4.8) at the endpoints of these curves take place.

To prove this we denote the segment of real trajectory by $\{y_1(t), \ldots, y_n(t)\}$ $(t_0 \leqslant t \leqslant t_1)$, put

$$h_s(t) = \varphi_s(t)(t - t_0)(t_1 - t)\alpha \qquad (s = 1, \ldots, k) \tag{4.9}$$

(functions $\varphi_s(t)$ are specified below), and take $h_{k+1}(t), \ldots, h_n(t)$ as the solutions of the system of differential equations

$$\dot{h}_i = e_i(y_1 + h_1, \ldots, y_k + h_k, \dot{y}_1 + \dot{h}_1, \ldots, \dot{y}_k + \dot{h}_k) - e_i(y_1, \ldots, y_k, \dot{y}_1, \ldots, y_k) \qquad (i = k+1, \ldots, n) \tag{4.10}$$

on the segment $[t_0, t_1]$ with the initial condition $h_{k+1}(t_0) = \ldots = h_n(t_0) = 0$.

If we differentiate (4.10) with respect to α we obtain

$$\frac{d}{dt}\delta q_i = \sum_{s=1}^{k} \left(\frac{\partial e_i}{\partial q_s} - \frac{d}{dt}\frac{\partial e_i}{\partial \dot{q}_s} \right) \delta q_s + \frac{d}{dt}\left(\sum_{s=1}^{k} \frac{\partial e_i}{\partial \dot{q}_s}\delta q_s \right) \qquad (i = k+1, \ldots, n), \tag{4.11}$$

where

$$\delta q_i = \left.\frac{\partial h_j}{\partial \alpha}\right|_{\alpha=0} \delta\alpha \qquad (j = 1, \ldots, n).$$

Then, taking into account (4.9), we derive that

$$\delta q_i(t_1) = \delta q_i(t_0) + \int_{t_0}^{t_1} \left[\sum_{s=1}^{k} \left(\frac{\partial e_i}{\partial q_s} - \frac{d}{dt}\frac{\partial e_i}{\partial \dot{q}_s} \right) \varphi_s \right] (t - t_0)(t_1 - t)\, dt \qquad (i = k+1, \ldots, n).$$

Choosing functions $\varphi_s(t)$ so that the expression in square brackets vanishes identically with respect to t, we obtain

$$\delta q_i(t_1) = \delta q_i(t_0) = 0 \qquad (i = k+1, \ldots, n).$$

Thus, we construct the family of curves $\{y_1(t) + h_1(t, \alpha), \ldots, y_n(t) + h_n(t, \alpha)\}$, $(t_0 \leqslant t \leqslant t_1)$ containing the segment M_0M_1 of the real motion. Curves of the family satisfy equations of constraints (3.16) and conditions (4.8) at the endpoints (M_0, t_0), (M_1, t_1). In the class of such curves we can correctly set the Lagrange conditional variational problem of stationary value of Hamilton action for the Chaplygin systems with constraints (3.16).

This general Lagrange problem for nonholonomic systems with constraints (3.14) was investigated by V. V. Rumyantsev [28]. With the help of undetermined multipliers $\mu_i(t)$, this problem of conditional extremum is reduced to the variational problem

$$\delta \int_{t_0}^{t_1} \left(L + \sum_{i=k+1}^{n} \mu_i f_i \right) dt = 0.$$

The equations of extremals are represented by the differential equations of the second order with respect to $q_j(t)$ and of the first order with respect to $\mu_i(t)$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_{i=k+1}^{n} \mu_i \left(\frac{\partial f_i}{\partial q_j} - \frac{d}{dt}\frac{\partial f_i}{\partial \dot{q}_j} \right) - \sum_{i=k+1}^{n} \dot{\mu}_i \frac{\partial f_i}{\partial \dot{q}_j} = 0 \qquad (j = 1, \ldots, n). \tag{4.12}$$

The general solution $\mathfrak{W}$ of system of equations (3.14), (4.12) depends on $2n$ arbitrary constants while the equations of motion of the nonholonomic system specify the $(2n - k)$-parametric family of trajectories.

We may ask what conditions for inclusion of a subset of these trajectories (particular solutions) or the whole family in the set $\mathfrak{W}$ have to be formulated. V. V. Rumyantsev has proved that such inclusion takes place if and only if for solutions of both systems the condition

$$\sum_{j=1}^{n}\sum_{i=k+1}^{n}\mu_i\left(\frac{\partial f_i}{\partial q_j}-\frac{d}{dt}\frac{\partial f_i}{\partial \dot q_j}\right)\delta q_j=0, \tag{4.13}$$

is fulfilled. Here $\delta q_1, \ldots, \delta q_n$ are virtual displacements specified with equations (3.15).

Condition (4.13) does not imply the integrability of constraints but it is severe enough. For example, for Chaplygin's systems with constraints (3.16), relation (4.13) is reduced to the equalities

$$\sum_{i=k+1}^{n}\mu_i\left(\frac{\partial e_i}{\partial q_s}-\frac{d}{dt}\frac{\partial e_i}{\partial \dot q_s}\right)=0 \qquad (s=1,\ldots,k),$$

which by virtue of the last $n-k$ equations of (4.12) can be written as

$$\sum_{i=k+1}^{n}\left(\frac{\partial T}{\partial \dot q_i}-p_{i0}+\mu_{i0}\right)\left(\frac{\partial e_i}{\partial q_s}-\frac{d}{dt}\frac{\partial e_i}{\partial \dot q_s}\right)=0 \qquad (s=1,\ldots,k). \tag{4.14}$$

Here p_{i0} are the initial values of impulses $\frac{\partial T}{\partial \dot q_i}$, μ_{i0} are the initial values of the Lagrange multipliers μ_i $(i=k+1,\ldots,n)$.

When assigning $\mu_{i0}=p_{i0}$ $(i=k+1,\ldots,n)$ we find that Chaplygin equations (3.17) have the Lagrange form for solutions of (4.12) and (3.16). If

$$\frac{\partial T}{\partial \dot q_i}=p_{i0} \qquad (i=k+1,\ldots,n)$$

are constants (general or particular) of the motion of holonomic system with Lagrangian L then assigning all $\mu_{i0}=0$ we satisfy (4.14) for the considered motion or class of motions.

Some generalizations of the principle of the least action and of some other variational principles for systems with nonholonomic constraints were studied by Hölder [65], V. S. Novoselov [19, 21], V. V. Rumyantsev [29] and others.

Russian translations of the original papers on differential and integral variational principles of mechanics may be found in collection [22]. The good review of variational principles of mechanics is presented in book [47].

5. Generalization of the Hamilton–Jacobi theorem

Since the equations of motion of systems with nonintegrable constraints can not be generally reduced to the Lagrange and Hamilton equations, the Hamilton–Jacobi theorem known for the holonomic systems (just as the Jacobi method) can not be generalized directly for the nonholonomic systems. There were several attempts of the generalization of this theorem. Some of them turn out to be erroneous, others are correct under severe restrictions that devaluate the practical significance of these generalizations. The bibliography of the problem and the analysis of some papers may be found in review [33].

Let's consider a generalization of the Hamilton–Jacobi theorem for systems with finite constraints in redundant generalized coordinates presented by G K. Suslov [34, 36]. It was generalized for nonholonomic systems by V. V. Rumyantsev [28, 30].

Introduce the generalized impulses by formulae

$$\pi_j = \frac{\partial T}{\partial \dot{q}_j} + \sum_{i=k+1}^{n} \mu_i \frac{\partial f_i}{\partial \dot{q}_i} \qquad (j = 1, \ldots, n), \tag{5.1}$$

where μ_i are multipliers of constraints (3.14). The generalized Hamilton function

$$H_1 = \sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j - L + \sum_{j=1}^{n} \sum_{i=k+1}^{n} \mu_i \frac{\partial f_i}{\partial \dot{q}_j}\dot{q}_j$$

is expressed with the help of (5.1) and (3.14) in the canonical variables q_j, π_j. The generalized Hamilton–Jacobi equation

$$\frac{\partial V}{\partial t} + H_1\left(t, q, \frac{\partial V}{\partial q}\right) = 0 \tag{5.2}$$

is the partial differential equation of the first order.

For (5.2) equations of characteristics have the form of canonical equations

$$\frac{dq_j}{dt} = \frac{\partial H_1}{\partial \pi_j}, \qquad \frac{d\pi_j}{dt} = -\frac{\partial H_1}{\partial q_j} \qquad (j = 1, \ldots, n). \tag{5.3}$$

According to the Jacobi theorem, the relations

$$\frac{\partial V}{\partial q_j} = \pi_j, \qquad \frac{\partial V}{\partial \alpha_j} = \beta_j \qquad (j = 1, \ldots, n)$$

are $2n$ integrals of equations (5.3) if $V(t, q, \alpha)$ is the complete integral of equation (5.2), α_j and β_j are arbitrary constants.

V. V. Rumyantsev proved that the solution of equations (5.3) represents the motion of the nonholonomic system with constraints (3.14) if and only if it satisfies condition (4.13), i. e. when Hamilton principle (4.6) has the character of the principle of stationary action.

This research was supported by "Leading Scientific Schools" (Grant 00–15–96150) and Russian Federal Program "Integration".

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