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On Hidden Ignorable Coordinates of Conservative Holonomic Systems with Three Degrees of Freedom

by

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Consider the Lagrange function

(1) $$L = 1/2\, a_{ij}\, \dot{q}^i \dot{q}^j + U \qquad (i, j = 1, 2, 3)$$

of any conservative holonomic system with three degrees of freedom and let $\mathrm{d}U \neq 0$ in a neighborhood W of a point $p \in M$ (M is the configuration manifold of the system). May we choose (at least in a small neighborhood W) the generalized coordinates of the system so that there will be cyclic (ignorable) ones among them? Here we give the invariant analytic indications of the existence of hidden ignorable coordinates.

Consider the family of equipotential surfaces $U = \mathrm{const}$ in W and let $\mathbf{e}_1(q)$, $\mathbf{e}_2(q)$ denote the unit vectors tangent to the lines of curvature of the corresponding surface $U = U(q)$ at any point q. In the case of parallel equipotential surface, i.e. when $a^{ij} U_i U_j$ is $g(U)$, the vector-functions $\mathbf{e}_1(q)$, $\mathbf{e}_2(q)$ are single-valued if and only if [1]

(2) $$(a^{ij} U_{ij})^2 - 2\epsilon^{ijk} \epsilon^{lmv} a_{il} U_{jm} U_{kv} \neq 0$$

at the point q. Here $\| a^{ij} \| = \| a_{ij} \|^{-1}$, ϵ^{ijk} is the Levi-Civita tensor [2], $U_i = \partial U / \partial q^i$, U_{jm} are the second covariant derivatives of U with respect to the Riemannian connection associated with the metric $\mathrm{d}s^2 = a_{ij}\, \mathrm{d}q^i\, \mathrm{d}q^j$. The standard convention ou summation is implied.

THEOREM 1. *In the small neighborhood W of any point $p \in M$, for which the relation (2) is true and $\mathrm{d}U \neq 0$, there exist two simultaneously ignorable generalized coordinates of the system if and only if five scalar invariants*

(3) $$\Delta_1 U = a^{ij} U_i U_j, \quad \pi_{\alpha\beta} = \mathbf{e}_\alpha \operatorname{rot} \mathbf{e}_\beta \qquad (\alpha, \beta = 1, 2)$$

depend on U only.

As far as the invariants (3) are expressed by the coefficients of (1) and their derivatives, then the necessity condition of the theorem follows from the definition of an ignorable coordinate. Sufficiency is proved [3] by introducing semigeodesic coordinate system $v^1, v^2, v^3 = f(U)$ is which the components of the vectors $\mathbf{e}_1(w_{11}, w_{12}, 0)$, $\mathbf{e}_2(w_{21}, w_{22}, 0)$ satisfy the equations

$$\frac{\partial w_{\lambda\alpha}}{\partial v^3} = \mathrm{e}^{\iota\beta}\pi_{\iota\lambda} w_{\beta\alpha} (\mathrm{e}^{11} = \mathrm{e}^{22} = 0, \mathrm{e}^{21} = -\mathrm{e}^{12} = 1; \alpha, \beta, \lambda, \iota = 1, 2)$$

with the initial condition

$$w_{11} = w_{22} = 1, \quad w_{12} = w_{21} = 0 \quad (\text{when} \quad v^3 = f \circ U(p))$$

and the metric is

$$ds^2 = (w_{12}^2 + w_{22}^2)(dv^1)^2 + 2(w_{11}w_{12} + w_{21}w_{22})dv^1 dv^2 + (w_{11}^2 + w_{21}^2)(dv^2)^2.$$

THEOREM 2. *Let the relation* (2) *be broken at every point of a neighborhood* $W \ni p$ *but* $dU \neq 0$. *Then there exist two simultaneously ignorable coordinates of the holonomic system if and only if the differential parameters*

$$(4) \qquad \Delta_1 U, \qquad \Delta_2 U = a^{ij} U_{ij}$$

depend on U *only and the absolute (Gaussian) curvature* K *of the surface* $U = U(p)$ *is zero, i.e.*

$$(5) \qquad \left[\frac{1}{2(\Delta_1 U)^2} \epsilon^{ijk} \epsilon^{lmr} U_i U_l U_{jm} U_{kr}\right]_{U = U(p)} = 0.$$

See the proof in [3]. In addition, if (5) is broken but $K = F(U)$, then one ignorable coordinate exists.

Let us assume that two among the invariants (3), (4), K and U are functionally independent. Let us denote them by x and y.

THEOREM 3. *If all the scalar invariants*

$$(6) \qquad \Delta_1 x, \qquad \Delta_1 y, \qquad \nabla(x, y) = a^{ij} x_i y_j$$

$$\operatorname{grad} x \cdot \operatorname{rot} \tau, \qquad \operatorname{grad} y \cdot \operatorname{rot} \tau, \qquad \tau \cdot \operatorname{rot} \tau$$

(*where* $\tau = \operatorname{grad} x \times \operatorname{grad} y$) *are functions of* x *and* y *only, then there exists a single ignorable generalized coordinate of the holonomic system. When there are three functionally independent invariants among* (3), (4), (6), K *and* U *the system has no hidden ignorable coordinates.*

The ignorable coordinate Q mentioned above is defined by solving the equation [3]

$$\operatorname{grad} Q(q^1, q^2, q^3) = (h\varphi + \tau) g$$

where g and h are some functions of x, y and

$$\varphi = \Delta_1 x \operatorname{grad} y - \nabla(x, y) \operatorname{grad} x .$$

Corresponding position coordinates of the holonomic system are for example x and y.

These results may be generalized in various directions.

References

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Abstract. *The invariant analytic indications of the existence of two, one and no one hidden ignorable generalized coordinates are formulated.*

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