ISSN 1028-3358, Doklady Physics, 2007, Vol. 52, No. 9, pp. 507–509. © Pleiades Publishing, Ltd., 2007.
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On Relative Equilibria of an Orbital Station in Regions near the Triangular Libration Points

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Presented by Academician V.V. Rumyantsev December 27, 2006

Received January 24, 2007

PACS numbers: 45.20.da, 95.40.+s

DOI: 10.1134/S1028335807090133

STATEMENT OF THE PROBLEM AND EQUATIONS OF EQUILIBRIUM

Consider the problem of motion of a mechanical system consisting of two bodies E and L traveling in Kepler's circular orbits under the action of forces of mutual attraction (Fig. 1). It is supposed that dimensions of body L are finite and body S is attached to the surface of body L at points P_1 and P_2 by a pair of inextensible weightless tethers of lengths l_1 and l_2, respectively (compare with [1–4]). Body S is assumed to be sufficiently light and to exert no influence on the motion of bodies E and L.

Choose a coordinate frame Oxy with the origin at the center of mass of the system, axis Ox directed from body E to body L, and axis Oy directed perpendicular to axis Ox in the plane of the orbit. This coordinate frame uniformly rotates about point O. If the total mass of points E and L is set equal to unity, as well as the distance between them, then point E with coordinates $(-\mu, 0)$ and point L $(1 - \mu, 0)$ have masses $1 - \mu$ and μ, respectively. The angular velocity of the uniform rotation of the coordinate frame is also set equal to unity. Denote the coordinates of points P_i $(i = 1, 2)$ by (a_i, b_i) and the coordinates of body S by (x, y).

Acted upon by Newton's forces exerted by E and L, centrifugal force, and the force of tension of the tethers, body S can be in equilibrium with respect to the rotating system. The potential of Newton's forces and centrifugal forces acting upon this body has the form

$$U = U_N + U_C, \tag{1}$$

where

$$U_N = -\left(\frac{1-\mu}{r_A} + \frac{\mu}{r_B}\right), \quad U_C = -\frac{1}{2}(x^2 + y^2),$$

$$r_A = \sqrt{(x+\mu)^2 + y^2}, \quad r_B = \sqrt{(x-1+\mu)^2 + y^2}.$$

By virtue of the inextensibility of the tether, the system obeys the constraints

$$f_i = (x - a_i)^2 + (y - b_i)^2 - l_i^2 = 0. \tag{2}$$

The equilibrium configurations of the system are the critical points of the Routh function

$$W = U + \frac{\lambda_1}{2} f_1 + \frac{\lambda_2}{2} f_2. \tag{3}$$

Equations determining the critical points have the form

$$\begin{aligned} \frac{\partial W}{\partial x} &= -F_x + \lambda_1(x - a_1) + \lambda_2(x - a_2) = 0, \\ \frac{\partial W}{\partial y} &= -F_y + \lambda_1(y - b_1) + \lambda_2(y - b_2) = 0. \end{aligned} \tag{4}$$

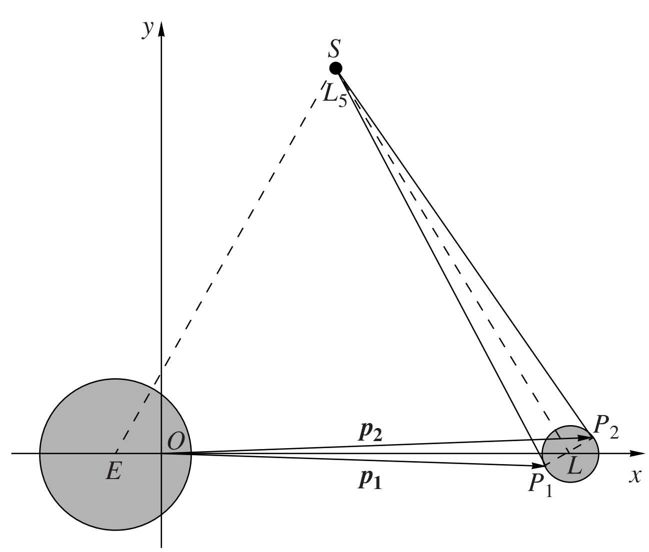

Fig. 1.

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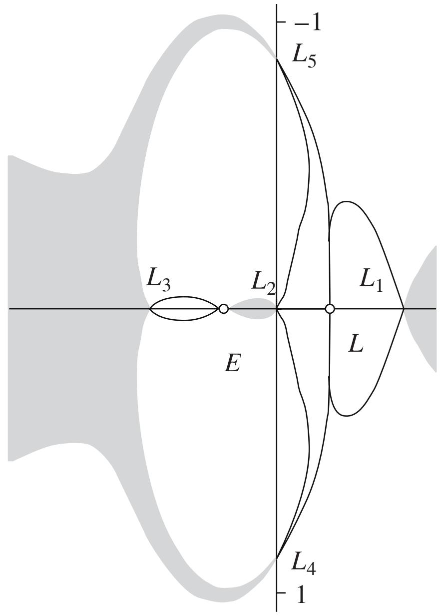

Fig. 2.

$$F_x = -\left(\frac{(1-\mu)(x+\mu)}{r_a^3} + \frac{\mu(x-1+\mu)}{r_b^3} - x\right),$$

$$F_y = -y\left(\frac{1-\mu}{r_a^3} + \frac{\mu}{r_b^3} - 1\right).$$

In the framework of the direct problem, Eqs. (4) together with constraint equations (2) constitute a system of four equations in four unknowns x, y, λ_1, and λ_2.

Let us solve Eqs. (4) with respect to λ_1 and λ_2. We have

$$\lambda_1 = \frac{F_x(y-b_2) - F_y(x-a_2)}{\Delta} = \Lambda_1(x, y),$$
$$\lambda_2 = \frac{F_y(x-a_1) - F_x(y-b_1)}{\Delta} = \Lambda_2(x, y), \quad (5)$$
$$\Delta = (x-a_1)(y-b_2) - (y-b_1)(x-a_2).$$

Since we consider solutions where both tethers are strained, the following conditions should hold:

$$\Lambda_1(x, y) > 0, \quad \Lambda_2(x, y) > 0. \quad (6)$$

Regions of the validity of conditions (6) are bounded by the curves

$$\Gamma_i = \{(x, y): \Lambda_i(x, y) = 0,\ i = 1, 2\},$$

which go through all five libration points of the restricted three-body problem, in particular, through the triangular libration points

$$L_4 = \left(\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2}\right), \quad L_5 = \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right).$$

Tangent lines to curves Γ_1 and Γ_2 at libration point L_5 are specified by the equations

$$\Gamma_1\colon \left(\mu^2 - \mu - \frac{1}{2}a_2 + \frac{\sqrt{3}}{6}b_2 + \mu a_2\right)\left(x - \frac{1}{2} + \mu\right) + \left(\frac{1}{2}b_2 - \frac{\sqrt{3}}{2}a_2 - b_2\mu\right)\left(y - \frac{\sqrt{3}}{2}\right) = 0,$$

$$\Gamma_2\colon \left(\mu + \frac{1}{2}a_1 - \frac{\sqrt{3}}{6}b_1 + \mu^2 - \mu a_1\right)\left(x - \frac{1}{2} + \mu\right) + \left(-\frac{1}{2}b_1 + \frac{\sqrt{3}}{2}a_1 + b_1\mu\right)\left(y - \frac{\sqrt{3}}{2}\right) = 0, \quad (7)$$

respectively.

Figure 2 shows curves Γ_1 and Γ_2 in the case where $\mu = 1/2$. These curves bound shaded regions where both tethers are strained. For small values of μ, which correspond to systems in the real world (e.g., $\mu = 0.00094$ for the Jupiter–Sun system or $\mu = 0.01215$ for the Earth–Moon system), these regions near the triangular libration points are rather narrow and, therefore, hard to demonstrate. However, they are nonempty and so it is expedient to consider the problem of exploiting them.

CONDITIONS OF REALIZABILITY OF THE SUSPENSION

Analyzing the shape of the shaded region, where both tethers are strained, near points L_4 and L_5 in Fig. 2 (for definiteness, in what follows we consider point L_5), one can easily see that the best case corresponds to the maximum angle between the tangent lines (and, hence, between the normals to them). Let us call it the noncollinearity condition and choose points at which the tethers are attached to the Moon's surface in such a way that this condition be satisfied. If

$$\mathbf{A}_0 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathbf{b}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the normals $\mathbf{n}_i$ ($i = 1, 2$) to the tangent lines can be represented as

$$\mathbf{n}_1 = \begin{pmatrix} -\frac{1}{2}a_2 + \frac{\sqrt{3}}{6}b_2 + \mu a_2 - \mu + \mu^2 \\ -\frac{\sqrt{3}}{2}a_2 + \frac{1}{2}b_2 - \mu b_2 \end{pmatrix} = \mathbf{A}_0\mathbf{p}_2 + \mu\mathbf{A}_1\mathbf{p}_2 + \mu\mathbf{b}_1 + \mu^2\mathbf{b}_2,$$

$$\mathbf{n}_2 = -\begin{pmatrix} \frac{1}{2}a_1 - \frac{\sqrt{3}}{6}b_1 - \mu a_1 + \mu - \mu^2 \\ \frac{\sqrt{3}}{2}a_1 - \frac{1}{2}b_1 + \mu b_1 \end{pmatrix}$$

$$= \mathbf{A}_0\mathbf{p}_1 + \mu\mathbf{A}_1\mathbf{p}_1 + \mu\mathbf{b}_1 + \mu^2\mathbf{b}_2.$$

The noncollinearity condition can be represented in terms of the cross product as

$$[\mathbf{n}_1, \mathbf{n}_2] \neq 0. \tag{8}$$

The columns of matrix $\mathbf{A}_0$ are linearly dependent, and so the matrix can be written in the form

$$\mathbf{A}_0 = (\mathbf{e}, \alpha\mathbf{e}), \quad \mathbf{e} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^T, \quad \alpha = -\frac{\sqrt{3}}{3}.$$

Therefore, one can use a uniform representation for the normals to tangent lines (7):

$$\mathbf{n}_i = \begin{pmatrix} c_i \\ d_i \end{pmatrix} = (a_i + \alpha b_i)\mathbf{e} + \mu\begin{pmatrix} a_i - 1 + \mu \\ -b_i \end{pmatrix}, \tag{9}$$

$$i = 1, 2.$$

Staying within the framework of the chosen scale of dimensional quantities, we denote the radius of the Moon by $\varepsilon \ll 1$. This quantity (which has the value of 0.0045) is of the same order of smallness as μ. Then the points at which the tethers are attached to its surface have the coordinates

$$a_i = 1 - \mu + \varepsilon\alpha_i, \quad b_i = \varepsilon\beta_i, \quad i = 1, 2, \tag{10}$$

where α_i and β_i are the coordinates of the attachment points in the reference system having the origin at the center of the Moon; these coordinates are normalized in such a way that the identity $\alpha_i^2 + \beta_i^2 = 1$ holds. In this case, expression (9) takes the form

$$\mathbf{n}_i = \left(1 - \mu + \varepsilon\left(\alpha_i - \frac{\beta_i}{\sqrt{3}}\right)\right)\mathbf{e} + \mu\varepsilon\boldsymbol{\gamma}_i, \quad \boldsymbol{\gamma}_i = \begin{pmatrix} \alpha_i \\ -\beta_i \end{pmatrix},$$

$$i = 1, 2,$$

and the cross product in condition (8) can be written as

$$[\mathbf{n}_1, \mathbf{n}_2] = \left[\left(1 - \mu + \varepsilon\left(\alpha_1 - \frac{\beta_1}{\sqrt{3}}\right)\right)\mathbf{e} + \mu\varepsilon\boldsymbol{\gamma}_1,\right.$$

$$\left.\left(1 - \mu + \varepsilon\left(\alpha_2 - \frac{\beta_2}{\sqrt{3}}\right)\right)\mathbf{e} + \mu\varepsilon\boldsymbol{\gamma}_2\right]$$

$$= \left(1 - \mu + \varepsilon\left(\alpha_1 - \frac{\beta_1}{\sqrt{3}}\right)\right)\mu\varepsilon[\mathbf{e}, \boldsymbol{\gamma}_2]$$

$$+ \left(1 - \mu + \varepsilon\left(\alpha_2 - \frac{\beta_2}{\sqrt{3}}\right)\right)\mu\varepsilon[\boldsymbol{\gamma}_1, \mathbf{e}] + \mu^2\varepsilon^2[\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2]$$

$$= \mu\varepsilon[\mathbf{e}, \boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_1] + \boldsymbol{o}(\mu\varepsilon).$$

Here, symbol $\boldsymbol{o}(\mu\varepsilon)$ denotes all terms of degree higher than two in both small parameters μ and ε. Thus, the condition of the best realizability of the suspension of body S near point L_5 (and L_4) can be written, up to terms of higher orders, in the form

$$[\mathbf{e}, \boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_1] \neq \mathbf{0}$$

or, in scalar form (since the cross product is perpendicular to the plane of motion of the bodies in the problem under consideration), as

$$\frac{\sqrt{3}}{2}(\alpha_2 - \alpha_1) + \frac{1}{2}(\beta_2 - \beta_1) \neq 0.$$

Since the left-hand side of this inequality contains the inner product of vector $\mathbf{q}_2 - \mathbf{q}_1$ ($\mathbf{q}_i = (\alpha_i, \beta_i)^T$, $i = 1, 2$) and the unit vector $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T$, this quantity attains its maximum exactly when the factors are collinear and vector $\mathbf{q}_2 - \mathbf{q}_1$ has the maximum length, which is equal to the diameter of the Moon. One can easily see that, in this case, the aforementioned diameter should be perpendicular to the line LC.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project nos. 05-01-00454-a and 06-01-90505-BNTS-a) and by the Council of the President of the Russian Federation for the Support of Young Russian Scientists and Leading Scientific Schools (project no. NSh-6667.2006.1).

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Translated by A. Pankrat'ev