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Bourov, A.; Chevallier, D.

On Routh Reduction and its Application in Rigid Body Dynamics

Routh reduction is a standard process reducing the system of Lagrange equations in generalized coordinates to a lower dimension system by use of first integrals corresponding to cyclic coordinates (see [1, 2]*). Here we demonstrate how this reduction can be performed for the Lagrange-Poincaré system describing the motion of a rigid body about a fixed point written with dependent coordinates and nonholonomic velocities. Some examples from a rigid body dynamics are considered. The idea of this method arises to the paper of* Lyapunov [3]*. The theory of the Routh reduction for the systems described by equations involving non-holonomic coordinates was developed in* [4, 5]*. The development of the approach of* Lyapunov *was done in* [6].

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1. Equations of motion

Some problems of the dynamics of a rigid body about a fixed point, in particular those of section 3 below, may be described by a system of equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \omega} \times \omega + \frac{\partial L}{\partial \alpha} \times \alpha + \frac{\partial L}{\partial \beta} \times \beta + \frac{\partial L}{\partial \gamma} \times \gamma\,, \tag{1}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \alpha \times \omega\,, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}t} = \beta \times \omega\,, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma \times \omega\,. \tag{2}$$

Here, α, β, γ are unit vectors directed along the axes of an absolute frame $NX_1X_2X_3$, ω is a vector of the body absolute angular velocity, $L = L(\omega, \alpha, \beta, \gamma)$ is a Lagrange function. All the vectors are defined by their coordinates with respect to a body fixed frame $Cx_1x_2x_3$.

These equations admit a Painlevé-Jacobi-energy integral,

$$\mathscr{I}_0 = H(\omega, \alpha, \beta, \gamma) = \left(\frac{\partial L}{\partial \omega}, \omega\right) - L - h = 0\,, \tag{3}$$

and six “geometric” integrals expressing the orthonormality of the vectors α, β, γ:

$$\mathscr{I}_\alpha = (\alpha, \alpha) - 1 = 0\,, \qquad \mathscr{I}_\beta = (\beta, \beta) - 1 = 0\,, \qquad \mathscr{I}_\gamma = (\gamma, \gamma) - 1 = 0\,, \tag{4}$$

$$\mathscr{I}_{\alpha\beta} = (\alpha, \beta) = 0\,, \qquad \mathscr{I}_{\beta\gamma} = (\beta, \gamma) = 0\,, \qquad \mathscr{I}_{\gamma\alpha} = (\gamma, \alpha) = 0\,. \tag{5}$$

Let us consider a new frame $NX'_1X'_2X'_3$ deduced from the initial frame by a rotation by the angle ψ about the axis NX_3 with unit vectors α', β', γ', such that

$$\alpha' = \alpha \cos\psi + \beta \sin\psi\,, \qquad \beta' = -\alpha \sin\psi + \beta\cos\psi\,, \qquad \gamma' = \gamma\,. \tag{6}$$

Then the angular velocity ω can be represented as

$$\omega = \Omega + \dot\psi\gamma\,, \tag{7}$$

where $\dot\psi$ is the angular velocity of the rotating frame $NX'_1X'_2X'_3$, and Ω is the relative angular velocity of the body with respect to the rotating frame.

Let us suppose that the Lagrange function is independent of α and β, i.e., the group of rotation about the axis NX_3 in the absolute space is the symmetry group. Then the equations of motion read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \omega} \times \omega + \frac{\partial L}{\partial \gamma} \times \gamma\,, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma \times \omega\,, \qquad L = L(\omega, \gamma)\,. \tag{8}$$

Besides integrals $\mathscr{I}_0$ and $\mathscr{I}_\gamma$ these equations admit a first integral

$$\mathscr{I}_1 = \left(\frac{\partial L}{\partial \omega}, \gamma\right) - p_\psi = 0\,, \tag{9}$$

expressing the rotational symmetry mentioned above. For mechanical systems this integral is linear with respect to the angular velocity.

This symmetry means that the change of variables $(\alpha, \beta, \gamma) \to (\alpha', \beta', \gamma')$ gives a new Lagrange function

$$\Lambda(\Omega, \dot\psi, \gamma) = L(\Omega + \dot\psi\gamma, \gamma) \tag{10}$$

which is independent of ψ. The first integral $\mathscr{I}_1$ reads

$$\mathscr{I}_1 = \frac{\partial \Lambda}{\partial \dot\psi} - p_\psi = 0\,. \tag{11}$$

Let us fix a level manifold $\{\mathscr{I}_1 = 0\}$ of the integral (9), and let us solve the corresponding equation with respect to $\dot\psi$,

$$\dot\psi = \dot\psi(\Omega,\ \gamma,\ p_\psi) \Leftrightarrow \left(\frac{\partial L(\Omega + \dot\psi(\Omega,\ \gamma,\ p_\psi)\,\gamma,\ \gamma)}{\partial \omega},\ \gamma\right) \equiv p_\psi\,. \tag{12}$$

We have the following theorem.

Theorem 1: *In the new variables* $(\Omega,\ \gamma)$ *with* $\dot\psi$ *chosen as in* (12), *the equations of motion* (8) *read*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \Omega} = \frac{\partial R}{\partial \Omega} \times \Omega + \frac{\partial R}{\partial \gamma} \times \gamma\,, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma \times \Omega\,, \tag{13}$$

where the Routh function R is defined by

$$R(\Omega,\ \gamma,\ p_\psi) = \left[\Lambda(\Omega,\ \dot\psi,\ \gamma) - \dot\psi\ \frac{\partial \Lambda(\Omega,\ \dot\psi,\ \gamma)}{\partial \dot\psi}\right]_{\dot\psi = \dot\psi(\Omega,\gamma,p_\psi)}. \tag{14}$$

After integration of these equations one can find the rotation of the rotating frame via integration of the equations

$$\dot\psi = -\frac{\partial R}{\partial p_\psi}\,. \tag{15}$$

At first glance this theorem does not bring any advantage because the obtained system has the same structure as the initial system of Lagrange equations. However, we have the following result:

Theorem 2: *Equations* (13) *admit the Painlevé-Jacobi-energy integral*

$$\mathscr{I}_0' = \left(\frac{\partial R}{\partial \Omega},\ \Omega\right) - R \tag{16}$$

coinciding with the integral (3) *and the geometric integral* $\mathscr{I}_\gamma$. *The integral*

$$\mathscr{I}_1' = \left(\frac{\partial R}{\partial \Omega},\ \gamma\right) \tag{17}$$

turns out to be trivial and equal to the constant p_ψ.

The proofs of both theorems are based on direct calculations given by the formulae

$$\frac{\partial R}{\partial \Omega_i} = \sum_j \frac{\partial L}{\partial \omega_j}\left(\delta_{ij} + \frac{\partial \dot\psi}{\partial \Omega_i}\ \gamma_j\right) - p_\psi\ \frac{\partial \dot\psi}{\partial \Omega_i} = \frac{\partial L}{\partial \omega_i} + \frac{\partial \dot\psi}{\partial \Omega_i}\left[\left(\frac{\partial L}{\partial \omega},\ \gamma\right) - p_\psi\right] \equiv \frac{\partial L}{\partial \omega_i}\,, \tag{18}$$

$$\frac{\partial R}{\partial \gamma_i} = \sum_j \frac{\partial L}{\partial \omega_j}\left(\dot\psi\delta_{ij} + \frac{\partial \dot\psi}{\partial \gamma_i}\ \gamma_j\right) + \frac{\partial L}{\partial \gamma_i} - p_\psi\ \frac{\partial \dot\psi}{\partial \gamma_i} = \frac{\partial L}{\partial \gamma_i} + \frac{\partial \dot\psi}{\partial \gamma_i}\left[\left(\frac{\partial L}{\partial \omega},\ \gamma\right) - p_\psi\right] + \dot\psi\ \frac{\partial L}{\partial \omega_i} \equiv \frac{\partial L}{\partial \gamma_i} + \dot\psi\ \frac{\partial R}{\partial \Omega_i}\,. \tag{19}$$

It is noteworthy that, in the general case, the frame $NX_1'X_2'X_3'$ does not rotate uniformly, the angular velocity of this rotation depends on the configuration of the system.

2. Steady motions

The advantage of the reduced system can be also pointed off in the determination of steady motions and in the investigation of their stability. According to Routh's procedure steady motions can be found as critical points of the Painlevé-Jacobi-energy integral considered as a function on a joint level of the other integrals. An appropriate Routh function is

$$W_\lambda(\omega,\ \gamma,\ \lambda,\ \mu) = \mathscr{I}_0 + \lambda\mathscr{I}_1 + \frac{\mu}{2}\ \mathscr{I}_\gamma\,, \tag{20}$$

where λ and μ are Lagrange multipliers. After simple evaluations, equations for steady motions read

$$\frac{\partial W_\lambda}{\partial \omega} = \frac{\partial^2 L}{\partial \omega^2}\ (\omega + \lambda\gamma) = 0\,, \tag{21}$$

$$\frac{\partial W_\lambda}{\partial \gamma} = \frac{\partial^2 L}{\partial \gamma\,\partial \omega}\ (\omega + \lambda\gamma) + \lambda\ \frac{\partial L}{\partial \omega} - \frac{\partial L}{\partial \gamma} + \mu\gamma = 0\,, \tag{22}$$

$$\frac{\partial W_\lambda}{\partial \lambda} = \frac{\partial W_\lambda}{\partial \mu} = 0\,. \tag{23}$$

In mechanics the Hessian of the Lagrange function computed with respect to the velocities is non-degenerate. Hence, equation (21) implies

$$\omega + \lambda\gamma = 0\,. \tag{24}$$

This means that the steady motions correspond to global rigid rotations (without change of configurations with respect to the rotating frame) about the axis γ with angular velocity $-\lambda$. Then equations (22) imply

$$\lambda\,\frac{\partial L(-\lambda\gamma,\,\gamma)}{\partial\omega} - \frac{\partial L(-\lambda\gamma,\,\gamma)}{\partial\gamma} + \mu\gamma = 0\,. \tag{25}$$

Deducing μ by the use of relation $(\gamma,\,\gamma) = 1$ from (23) and substituting we obtain the system

$$\lambda\,\frac{\partial L(-\lambda\gamma,\,\gamma)}{\partial\omega} - \frac{\partial L(-\lambda\gamma,\,\gamma)}{\partial\gamma} + \left[\left(\frac{\partial L(-\lambda\gamma,\,\gamma)}{\partial\gamma},\,\gamma\right) - \lambda p_\psi\right]\gamma = 0 \tag{26}$$

for the determination of the steady configurations with respect to the rotating frame. When the configuration is obtained from (26) the relation between the angular velocity and the constant p_ψ can be found from another integral in (23).

However, the same result can be deduced in a shorter way from the consideration of the reduced system. Indeed, consider a pencil of integrals of the reduced system

$$W_\mu(\Omega,\,\gamma,\,p_\psi,\,\mu) = \mathscr{I}_0' + \frac{\mu}{2}\,\mathscr{I}_\gamma \tag{27}$$

as Routh function. Its critical points can be found from equations

$$\frac{\partial W_\mu}{\partial\Omega} = \frac{\partial^2 R}{\partial\Omega^2}\,\Omega = 0\,, \tag{28}$$

$$\frac{\partial W_\mu}{\partial\gamma} = \frac{\partial^2 R}{\partial\gamma\,\partial\Omega}\,\Omega - \frac{\partial R}{\partial\gamma} + \mu\gamma = 0\,, \tag{29}$$

$$\frac{\partial W_\mu}{\partial\lambda} = \frac{\partial W_\mu}{\partial\mu} = 0\,. \tag{30}$$

For mechanical systems the Hessian of the Routh function computed with respect to relative velocities is nondegenerate. Hence equations (28) imply

$$\Omega = 0\,. \tag{31}$$

This means that on steady motions the mechanical system does not move with respect to the rotating frame which itself rotates about the axis γ with angular velocity $\dot\psi$. Then equation (29) implies

$$-\frac{\partial R(0,\,\gamma,\,p_\psi)}{\partial\gamma} + \mu\gamma = 0\,. \tag{32}$$

Eliminating μ by the use of (30) we obtain a system

$$-\frac{\partial R(0,\,\gamma,\,p_\psi)}{\partial\gamma} + \left(\frac{\partial R(0,\,\gamma,\,p_\psi)}{\partial\gamma},\,\gamma\right)\gamma = 0 \tag{33}$$

for the determination of steady configurations. Equalities (18) and (19) provide an equivalence of two approaches to the determination of steady motions. However, within the frame of the second approach the dependence of the angular velocity on the constant p_ψ is deduced from (15). The function

$$U_r(\gamma,\,p_\psi) = -R(0,\,\gamma,\,p_\psi) \tag{34}$$

is an *amended* potential of the rotating system. Thus the steady motions correspond to critical points of the amended potential considered as a function on a joint level of geometric integrals. This is a well-known theorem of Routh.

The rotation of the frame $NX_1'X_2'X_3'$ for steady motions is uniform.

3. Examples

Example 1: Motion of a rigid body about a fixed point in an axially symmetric force field

In mechanics of rigid bodies about a fixed point in an axially symmetric field motions of a system can be described in more general situation by the system (8) with the Lagrange function

$$L(\omega,\,\gamma) = \tfrac{1}{2}\,(I\omega,\,\omega) + (A(\gamma),\,\omega) - U(\gamma)\,. \tag{35}$$

The first term describes the kinetic energy of the body (or its modified form as for an apparent gyrostat). The second term generates gyroscopic forces. In particular, if $A(\gamma) = K = \text{const}$, we have a gyrostat of Kelvin (or the apparent gyrostat) – see, for example [7]. Though the vectors $A(\gamma) = B\gamma$ also appear in rigid body dynamics with various physical fields (see, for example, [8], pp. 32–33), one can use them as a source of controlling torques via a slight modification of the approach to control and stabilization proposed in [9]. This point is the matter of separate discussion. The last term is equal to the opposite of the potential energy of the system. The vector γ is directed along the axis of symmetry. Thus we have

$$\Lambda(\omega, \gamma, \dot\psi) = \frac{1}{2}\,(I(\Omega + \dot\psi\gamma),\, \Omega + \dot\psi\gamma) + (A(\gamma),\, \Omega + \dot\psi\gamma) - U(\gamma)$$
$$= \frac{1}{2}\,(I\Omega,\, \Omega) + (A(\gamma),\, \Omega) - U(\gamma) + \dot\psi[(I\Omega,\, \gamma) + (A(\gamma),\, \gamma)] + \frac{\dot\psi^2}{2}\,(I\gamma,\, \gamma) = \Lambda_0 + \Lambda_1 + \Lambda_2\,. \quad (36)$$

The integral (9) reads

$$(I\Omega,\, \gamma) + (A(\gamma),\, \gamma) + \dot\psi(I\gamma,\, \gamma) = p_\psi\,. \quad (37)$$

Resolving this equation with respect to $\dot\psi$ we obtain

$$\dot\psi = \frac{p_\psi - [(I\Omega,\, \gamma) + (A(\gamma),\, \gamma)]}{(I\gamma,\, \gamma)}\,. \quad (38)$$

Then the Routh function reads

$$R(\Omega, \gamma, p_\psi) = \frac{1}{2}\,(I\Omega,\, \Omega) + (A(\gamma),\, \Omega) - U(\gamma) - \frac{1}{2}\,\frac{(p_\psi - ((I\Omega,\, \gamma) + (A(\gamma),\, \gamma)))^2}{(I\gamma,\, \gamma)}\,. \quad (39)$$

The function

$$U_r = -R(0, \gamma, p_\psi) = \frac{1}{2}\,\frac{(p_\psi - (A(\gamma),\, \gamma))^2}{(I\gamma,\, \gamma)} + U(\gamma) \quad (40)$$

is exactly the well-known reduced (amended) potential for this class of problems.

Consider the quadratic part of the function (39) with respect to the relative angular velocity. It reads

$$R_2(\Omega, \gamma) = \frac{1}{2}\left((I\Omega,\, \Omega) - \frac{(I\Omega,\, \gamma)^2}{(I\gamma,\, \gamma)}\right). \quad (41)$$

In mechanics the tensor I is positive definite. Hence by virtue of the inequality of Cauchy-Schwartz we obtain

$$(I\Omega,\, \Omega)\,(I\gamma,\, \gamma) - (I\Omega,\, \gamma)^2 \geq 0\,, \quad (42)$$

which is valid for any Ω and γ the quadratic form (41) is also positive definite. This fact plays a decisive role in validation of the use of the reduced potential and the Routh theory for studying stability of steady motions of mechanical systems.

Example 2: Motion of a heavy rigid body on a horizontal plane without friction

Consider a motion of a heavy rigid body on a horizontal plane without friction. Suppose that the body is convex, i.e., it has only one point of a contact with the plane. Let the plane be given by the equation

$$X_3 = 0 \quad (43)$$

in the absolute frame $NX_1X_2X_3$. The Newtonian attraction is directed along the vector $-\gamma$. If in the body fixed frame $Cx_1x_2x_3$ (C is the center of mass) the equation of the boundary of the body is given by

$$f(x_1,\, x_2,\, x_3) = 0\,, \quad (44)$$

then the coordinates of the point of contact, $x = x(\gamma)$, can be found from the system of equations

$$-\gamma = \frac{\operatorname{grad} f(x)}{|\operatorname{grad} f(x)|} \quad (45)$$

expressing the collinearity of the vector γ and the external normal at the point of contact. Thus, it turns out that the distance from the center of mass to the plane is equal to

$$C_3(\gamma) = -(x(\gamma),\, \gamma)\,. \quad (46)$$

One can see that the system has two independent symmetries related to translations along the horizontal plane. Thus it is sufficient to suppose that the center of mass of the system moves only along the vertical.

Usually (cf. [2]) the dynamics of the system is described by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \omega} = \frac{\partial T}{\partial \omega} \times \omega + x(\gamma) \times \mathbf{R}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial V_3} = -mg + \mathscr{R}\,, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma \times \omega\,;$$
$$\mathbf{R} = \mathscr{R}\gamma\,, \qquad T = \tfrac{1}{2}\left(mV_3^2 + (I\omega,\, \omega)\right), \qquad V_3 = \dot{C}_3\,, \tag{47}$$

where $\mathbf{R}$ is the force of reaction of the plane, m is the mass of the body, I is the central tensor of inertia.

However, one can use a direct variational approach for describing the mechanics of such a system. From (46) and Poisson's equation one has

$$\dot{C}_3(\gamma) = \left(\frac{\partial C_3}{\partial \gamma},\, \dot{\gamma}\right) = \left(\frac{\partial C_3}{\partial \gamma},\, \gamma \times \omega\right). \tag{48}$$

Then the kinetic energy of the system can be expressed as

$$T = \frac{1}{2}\left(m\left(\frac{\partial C_3}{\partial \gamma},\, \gamma \times \omega\right)^2 + (I\omega,\, \omega)\right). \tag{49}$$

The potential energy is

$$U = -mgC_3(\gamma)\,, \tag{50}$$

and the equations of motion can be presented as (8) with the Lagrange function $L(\omega,\, \gamma) = T - U$. The substitution (7) shows that the first term in the expression for the kinetic energy, related to the vertical motion of the center of mass, does not depend on $\dot{\psi}$. Then after transformations one deduces the expression for the Routh function,

$$R(\Omega,\, \gamma,\, p_\psi) = \frac{1}{2}\left((I\Omega,\, \Omega) + m\left(\frac{\partial C_3}{\partial \gamma},\, \gamma \times \Omega\right)^2\right) - \frac{\left(p_\psi - (I\Omega,\, \gamma)\right)^2}{(I\gamma,\, \gamma)} - mgC_3(\gamma)\,. \tag{51}$$

All the arguments concerning the "reduced" kinetic energy are also valid.

Example 3: Motion of a rigid body in an ideal fluid

Consider a motion of a rigid body in an ideal incompressible fluid filling all of the space and at rest at infinity. Suppose that external forces and torques are absent. The dynamics of the system is described by the well-known Kirchhoff system (see, for example, [10])

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \omega} = \frac{\partial T}{\partial \omega} \times \omega + \frac{\partial T}{\partial v} \times v\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial v} = \frac{\partial T}{\partial v} \times \omega\,, \tag{52}$$

$$T(\omega,\, v) = \tfrac{1}{2}\left((a\omega,\, \omega) + 2(b\omega,\, v) + (cv,\, v)\right). \tag{53}$$

Here ω is the angular velocity of the body about the point C fixed in the body, v is the linear velocity of this point, a, b, and c are matrices related to the inertia tensor of the body and to the tensor of the added mass, T is the kinetic energy of the system. Obviously, the equations (52) do not coincide with the system (8) considered in this paper.

Besides of the energy integral

$$\mathscr{I}_0 = \left(\frac{\partial T}{\partial \omega},\, \omega\right) + \left(\frac{\partial T}{\partial v},\, v\right) - T \tag{54}$$

the system (52) admits two first integrals,

$$\mathscr{I}_1 = \left(\frac{\partial T}{\partial \omega},\, \frac{\partial T}{\partial v}\right) = p_\psi \tag{55}$$

and

$$\mathscr{I}_2 = \left(\frac{\partial T}{\partial v},\, \frac{\partial T}{\partial v}\right) = k^2\,, \qquad k > 0\,. \tag{56}$$

The latter integral is a consequence of an important property of the equations of motion. By virtue of the second subsystem of these equations the direction and the magnitude of the impulse vector

$$p = \frac{\partial T}{\partial v} = b\omega + cv \tag{57}$$

in the absolute space are conserved.

One can make a partial Legendre transformation with respect to the sole variable v. Then from (57) one has

$$v = c^{-1}p - c^{-1}b\omega\,, \tag{58}$$

and according to the Legendre transformation one has

$$L(\omega, p) = [T(\omega, v) - (p, v)]_{v=c^{-1}p-c^{-1}b\omega} = \frac{1}{2}[(a\omega, \omega) + 2(b\omega, c^{-1}p - c^{-1}b\omega) + (c(c^{-1}p - c^{-1}b\omega), c^{-1}p - c^{-1}b\omega)]$$
$$= \frac{1}{2}[(A\omega, \omega) + 2(B\omega, p) + (Cp, p)]. \tag{59}$$

The equations of motion may be written in the same form as (13)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \omega} = \frac{\partial T}{\partial \omega} \times \omega + \frac{\partial T}{\partial p} \times p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = p \times \omega. \tag{60}$$

The only difference to the above case lies in the value of the first integral $\mathscr{I}_2$. After a change of variables, $p = k\gamma$, we deduce exactly the system (13) with Lagrange function

$$L = \frac{1}{2}((\mathscr{A}\omega, \omega) + 2(\mathscr{B}\omega, \gamma) + (\mathscr{C}\gamma, \gamma)), \qquad \mathscr{A} = A, \quad \mathscr{B} = kB, \quad \mathscr{C} = k^2 C. \tag{61}$$

Then one can use all the calculations from the first example in order to obtain the Routh function, etc.

Concerning the latter example, one can say that all the steps are independent of the structure of the kinetic energy, the same steps are also valid for more complicated hydrodynamic systems. Similar arguments are valid for the hydrodynamic systems in a homogeneous field of Newtonian attraction if an Archimedian force is balanced by the Newtonian attraction. Following the classical papers by Chaplygin, written in the early twentieth century, recent publications bring the subject to the fore (see, for example, [11]).

4. Sufficient stability conditions

In order to obtain sufficient stability conditions we shall use again the methods due to Routh-Lyapunov-Chetayev. Let us consider the second variation of the function W_μ calculated for a given steady motion. It reads

$$2\delta^2 W_\mu = \left(\frac{\partial^2 R}{\partial \Omega^2}\,\delta\Omega, \delta\Omega\right) + \left(\left(\mu\mathbf{E} - \frac{\partial^2 R}{\partial \gamma^2}\right)\delta\gamma, \delta\gamma\right). \tag{62}$$

Let us consider its restriction to the linear manifold,

$$\delta\mathscr{I}_\gamma = \{\delta\gamma : (\gamma, \delta\gamma) = 0\}. \tag{63}$$

For mechanical systems the first term in (62) is always a positive definite quadratic form. Then the given steady motion is stable in the secular sense if the restriction of the second term, i.e. of the second variation of the amended potential, to the linear manifold (63) is also positive definite – see, for example, [12]. This means that the amended potential, considered as a function restricted to a joint level of geometric integrals, takes its strict minimum value on steady motions.

5. Necessary conditions of stability

Investigation of the restriction of the second variation of the amended potential to the linear manifold allows to determine not only sufficient conditions for stability but also the degree of instability. This degree of instability is equal to the number of negative eigenvalues of the matrix corresponding to the quadratic form obtained after restriction of the second variation of the amended potential to the linear manifold (63). If this number is odd then the steady motion is unstable. Otherwise one can expect a gyroscopic stabilization. The considered mechanical system has two degrees of freedom after Routh reduction. Hence the gyroscopic stabilization can take place only if the degree of instability is equal to 2.

In order to study the conditions of gyroscopic stabilization of Lagrangian systems one can introduce generalized Lagrangian coordinates, write down the equations of motion, find their solutions corresponding to steady motions, and linearize the equations of motion in the neighborhood of a steady motion. Then the equation for the eigenfrequences of the considered mechanical system reads

$$f(\lambda) = \det(\mathbf{A}\lambda^2 + 2\mathbf{G}\lambda + \mathbf{C}) = 0, \tag{64}$$

where $\mathbf{A}$ is a positive definite symmetric matrix obtained from the kinetic energy, $\mathbf{G}$ is a skew symmetric matrix of linearized gyroscopic forces, $\mathbf{C}$ is the symmetric matrix of the second variation of a generalized potential. There exist various sufficient conditions, which can be checked without calculation of eigenvalues, for gyroscopic stabilization to be possible or impossible [13, 14].

However, the reduced equations (13) linearized in a small neighborhood of a steady motion read

$$\frac{\partial^2 R}{\partial \Omega^2}\,\delta\dot{\Omega} = -\frac{\partial^2 R}{\partial \Omega\,\partial \gamma}\,\delta\dot{\gamma} + \left(\frac{\partial^2 R}{\partial \gamma\,\partial \Omega}\,\delta\Omega + \frac{\partial^2 R}{\partial \gamma^2}\,\delta\gamma\right) \times \gamma + \frac{\partial R}{\partial \Omega} \times \delta\Omega + \frac{\partial R}{\partial \gamma} \times \delta\gamma, \qquad \delta\dot{\gamma} = \gamma \times \delta\Omega. \tag{65}$$

At first glance an appropriate equation for the determination of eigenfrequences is far from (64). It turns out that this equation can be rewritten in the desirable form. In order to do this let us differentiate once the first equation (65) with respect to time and, after this, let us substitute it into the second one. We obtain

$$\frac{\partial^2 R}{\partial \Omega^2}\,\delta\ddot{\Omega} = -\frac{\partial^2 R}{\partial \Omega\,\partial \gamma}\,(\gamma \times \delta\dot{\Omega}) + \left(\frac{\partial^2 R}{\partial \gamma\,\partial \Omega}\,\delta\dot{\Omega} + \frac{\partial^2 R}{\partial \gamma^2}\,(\gamma \times \delta\Omega)\right) \times \gamma + \frac{\partial R}{\partial \Omega} \times \delta\dot{\Omega} + \frac{\partial R}{\partial \gamma} \times (\gamma \times \delta\Omega)\,.$$

Then, after appropriate calculations, an equation for the determination of eigenfrequences can be written as (64) where

$$\mathbf{A}_{ij} = \frac{\partial^2 R}{\partial \Omega_i\,\partial \Omega_j}\,, \qquad i,\, j = 1,\, 2,\, 3\,;$$

$$\mathbf{G}_{12} = \frac{1}{2}\left(-\frac{\partial^2 R}{\partial \Omega_1\,\partial \gamma_1}\,\gamma_3 - \frac{\partial^2 R}{\partial \Omega_2\,\partial \gamma_2}\,\gamma_3 + \frac{\partial^2 R}{\partial \Omega_1\,\partial \gamma_3}\,\gamma_1 + \frac{\partial^2 R}{\partial \Omega_2\,\partial \gamma_3}\,\gamma_2 + \frac{\partial R}{\partial \Omega_3}\right),$$

$$\mathbf{G}_{23} = \frac{1}{2}\left(-\frac{\partial^2 R}{\partial \Omega_2\,\partial \gamma_2}\,\gamma_1 - \frac{\partial^2 R}{\partial \Omega_3\,\partial \gamma_3}\,\gamma_1 + \frac{\partial^2 R}{\partial \Omega_2\,\partial \gamma_1}\,\gamma_2 + \frac{\partial^2 R}{\partial \Omega_3\,\partial \gamma_1}\,\gamma_3 + \frac{\partial R}{\partial \Omega_1}\right),$$

$$\mathbf{G}_{31} = \frac{1}{2}\left(-\frac{\partial^2 R}{\partial \Omega_3\,\partial \gamma_3}\,\gamma_2 - \frac{\partial^2 R}{\partial \Omega_1\,\partial \gamma_1}\,\gamma_2 + \frac{\partial^2 R}{\partial \Omega_3\,\partial \gamma_2}\,\gamma_3 + \frac{\partial^2 R}{\partial \Omega_1\,\partial \gamma_2}\,\gamma_1 + \frac{\partial R}{\partial \Omega_2}\right);$$

$$\mathbf{G}_{ij} = -\mathbf{G}_{ji}\,,$$

$$\mathbf{C}_{11} = \left(\frac{\partial R}{\partial \gamma},\, \gamma\right)(\gamma_2^2 + \gamma_3^2) - \left(\frac{\partial^2 R}{\partial \gamma_2^2}\,\gamma_3^2 - 2\,\frac{\partial^2 R}{\partial \gamma_2\,\partial \gamma_3}\,\gamma_2\gamma_3 + \frac{\partial^2 R}{\partial \gamma_3^2}\,\gamma_2^2\right),$$

$$\mathbf{C}_{22} = \left(\frac{\partial R}{\partial \gamma},\, \gamma\right)(\gamma_3^2 + \gamma_1^2) - \left(\frac{\partial^2 R}{\partial \gamma_3^2}\,\gamma_1^2 - 2\,\frac{\partial^2 R}{\partial \gamma_3\,\partial \gamma_1}\,\gamma_3\gamma_1 + \frac{\partial^2 R}{\partial \gamma_1^2}\,\gamma_3^2\right),$$

$$\mathbf{C}_{33} = \left(\frac{\partial R}{\partial \gamma},\, \gamma\right)(\gamma_1^2 + \gamma_2^2) - \left(\frac{\partial^2 R}{\partial \gamma_1^2}\,\gamma_2^2 - 2\,\frac{\partial^2 R}{\partial \gamma_1\,\partial \gamma_2}\,\gamma_1\gamma_2 + \frac{\partial^2 R}{\partial \gamma_2^2}\,\gamma_1^2\right),$$

$$\mathbf{C}_{12} = -\left(\frac{\partial R}{\partial \gamma},\, \gamma\right)\gamma_1\gamma_2 - \frac{\partial^2 R}{\partial \gamma_2\,\partial \gamma_3}\,\gamma_1\gamma_3 - \frac{\partial^2 R}{\partial \gamma_3\,\partial \gamma_1}\,\gamma_3\gamma_2 + \frac{\partial^2 R}{\partial \gamma_2\,\partial \gamma_1}\,\gamma_3^2 + \frac{\partial^2 R}{\partial \gamma_3^2}\,\gamma_1\gamma_2\,,$$

$$\mathbf{C}_{23} = -\left(\frac{\partial R}{\partial \gamma},\, \gamma\right)\gamma_2\gamma_3 - \frac{\partial^2 R}{\partial \gamma_3\,\partial \gamma_1}\,\gamma_2\gamma_1 - \frac{\partial^2 R}{\partial \gamma_1\,\partial \gamma_2}\,\gamma_1\gamma_3 + \frac{\partial^2 R}{\partial \gamma_3\,\partial \gamma_2}\,\gamma_1^2 + \frac{\partial^2 R}{\partial \gamma_1^2}\,\gamma_2\gamma_3\,,$$

$$\mathbf{C}_{31} = -\left(\frac{\partial R}{\partial \gamma},\, \gamma\right)\gamma_3\gamma_1 - \frac{\partial^2 R}{\partial \gamma_1\,\partial \gamma_2}\,\gamma_3\gamma_2 - \frac{\partial^2 R}{\partial \gamma_2\,\partial \gamma_3}\,\gamma_2\gamma_1 + \frac{\partial^2 R}{\partial \gamma_1\,\partial \gamma_3}\,\gamma_2^2 + \frac{\partial^2 R}{\partial \gamma_2^2}\,\gamma_3\gamma_1\,.$$

The obtained structure for the matrix **C** is due to the fact that, for the considered motions, the equality (33) holds.

6. Concluding remarks

The results on Routh reduction and on sufficient stability conditions can be easily extended to a system of rigid bodies moving in a three-dimensional euclidean space. However, a simple structure for the equation of the eigenvalues cannot be expected because of the interaction between the bodies. Moreover, the results can be extended to the situation when the system obeys nonholonomic constraints of special kind.

It would be also interesting and important to clarify the connection of the dynamics of the reduced system in nonholonomic variables with "strange" stability properties of systems subjected to gyroscopic forces expressed by nonholonomic coordinates [15].

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Addresses: Dr. Alexandre Bourov, Computing Centre of the Russian Academy of Sciences, Vavilova, 40, 117 967 Moscow, Russia; Prof. Dominique Chevallier, CERMICS ENPC, 6 et 8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, F-77455 Marne-la-Valle, France