Mechanical interpretation of negative and imaginary tension of a tether in a linear parallel force field

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In 1878, Appell provided a mechanical interpretation of the imaginary time in the simple pendulum problem. He also cited Poinsot for providing a mechanical interpretation of negative tether tension. Yet, posing a problem for identifying equilibria of a tether in a field of repelling forces whose magnitudes were proportional to the distance from a fixed axis, to which the tether was attached, Appell restricted the scope of solutions, being considered, only to those for which strict positivity of tension held. With this positive tension assumption being enforced, the stated problem provided an example of a problem with a countably infinite family of solutions.

Our aim is providing two dual mechanical interpretations of negative tension in the Appell-Pojaritsky tether equilibrium problem, one of which is Poinsots interpretation. Furthermore, a mechanical interpretation of imaginary tension (with two opposite signs) will, also, be given, yielding new solutions to the problem of tether equilibria in a linear parallel force field.

An introduction: Efficient calculation of elliptic integrals

Introduce a sequence of pairs $\{x_n,\ y_n\}_{n=0}^{\infty}$:

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := \sqrt{x_n y_n}.$$

Define *the arithmetic-geometric mean* of two positive numbers x and y as the (common) limit of the (descending) sequence $\{x_n\}_{n=1}^{\infty}$ and the (ascending) sequence $\{y_n\}_{n=1}^{\infty}$ with $x_0 = x$, $y_0 = y$.

Gauss [9] had discovered the identity:

$$\int_0^1 \frac{dx}{\sqrt{(1-(1-k^2)\,x^2)\,(1-x^2)}} = \frac{\pi}{2\,M(k)},\ 0 < k < 1, \tag{1}$$

where $M(x)$ is the arithmetic-geometric mean of 1 and x, enabling highly efficient calculation of complete elliptic integrals of the first kind.

The identity (1) holds true when k (strictly) exceeds 1. In particular, the lemniscate integral might be calculated up to four (ten) decimals, after two (three) iterations (for evaluating $M(\sqrt{2})$):

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2\,M(\sqrt{2})} \approx 1.311028777.$$

The reciprocal of $M(\sqrt{2})$ is called Gauss constant; Gauss, having calculated it to 11 decimal places, wrote in his diary [10], on May 30, 1799, that the discovery made "opens an entirely new field of analysis". Thereby, the beautiful field of elliptic functions and elliptic curves intertwining analysis, algebra and geometry was born. The formula given via equation (1) signified a qualitative transition in connecting the study of elliptic integrals of the first kind with studying elliptic functions.

An efficient algorithm for computing (incomplete) elliptic integrals of the first kind, based on inverting the doubling formula for points on an elliptic curve, is given in [5].

Introduce, next, a sequence of triples $\{x_n,\ y_n,\ z_n\}_{n=0}^{\infty}$:

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := z_n + \sqrt{(x_n - z_n)(y_n - z_n)},\ z_{n+1} := z_n - \sqrt{(x_n - z_n)(y_n - z_n)}.$$

Define *the modified arithmetic-geometric mean* of two positive numbers x and y as the (common) limit of the (descending) sequence $\{x_n\}_{n=1}^{\infty}$ and the (ascending) sequence $\{y_n\}_{n=1}^{\infty}$ with $x_0 = x$, $y_0 = y$ and $z_0 = 0$.

A formula analogous with (1) for calculating complete elliptic integrals of the second kind was discovered on December 16, 2011 while conducting fundamental research concerning Tethered Sattelite Systems at the Division of Stability Theory and Mechanics of Controlled Systems (CCRAS, Moscow, Russia) headed by S. Ya. Stepanov:

$$\int_0^1 \sqrt{\frac{1-(1-k^2)\,x^2}{1-x^2}}\;dx = \frac{\pi\,N(k^2)}{2\,M(k)},\ \ 0<k<1, \tag{2}$$

where $N(x)$ is the modified arithmetic-geometric mean of 1 and x.

The periods of a simple pendulum

Had Appell known of Gauss method (for calculating complete elliptic integrals of the first kind) he would not had had to "discover" a mechanical interpretation of the "imaginary period" [6] of a simple pendulum [7]. The (two-valued) period T[1] of a simple pendulum might be clearly and succinctly expressed as:

$$T = 2\pi c\,\sqrt{\frac{l}{g}},\ \ c := c(\theta) = \begin{cases} 1/M\,(\cos(\theta/2)) & : g>0 \\ \sqrt{-1}/M\,(\sin|\theta/2|) & : g<0 \end{cases}$$

where l is the length of the pendulum,
g is the acceleration (due to gravity),

θ is the angle of the maximal inclination from the pointing (in the positive direction) downwards vertical, as shown in Fig. 1, $0 < |\theta| < \pi$.

The configuration space of the pendulum upon which an external force, of constant magnitude and direction (presumably acting along the vertical), being exerted, is a circle. In other words, the weight of the pendulum is (holonomically) constrained to lie on a circle, so that its radial component is counterbalanced by pivot reaction force.

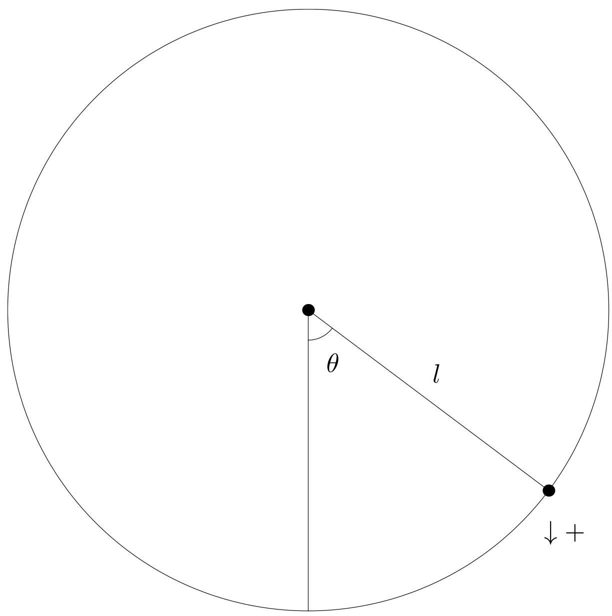


Fig. 1: The pendulum

The period corresponding to the (upper) value of c with g being positive correspond to gravity pointing downwards (as is customarily assumed). The complementary period, corresponding to the (lower) value of c with g being negative, is then readily seen to correspond to reversing the direction of gravity. Surprisingly too many (if not all!) "popular" references on elliptic functions such as [11, p. 59, 77], and "authoritative" references on mechanics such as [13, p. 73] have missed (up to this day) these elegant and powerful expressions (for which Gauss must be solely credited), routinely providing, instead, either unfinished calculations or cumbersome power series representation (lacking iterativity and convergence expediency), hardly enabling an understanding of the double-valuedness of T. A particular (self complementary) case, to be pointed out, corresponds to the (middle) value $\theta = \pi/2$, for which $|c| = \sqrt{2}/M(\sqrt{2})$. For a full appreciation of Gauss formula one must employ it for values of θ approaching π when traditional calculations of T via its series representation eventually fail to converge at any reasonable time!

[1] T regraded as a function of $|g|$ – the modulus of g. The choice of the positive direction along the "vertical" is, after all, arbitrarily made, so, regardless of the choice, both signs of g must be accounted for.

Tether equilibria in a parallel force field

A tether, being in equilibrium in a parallel force filed, lies in a plane parallel to the lines of force. In this plane we impose an orthogonal coordinate system Oxz with the "vertical" axis Oz parallel to the lines of force. The projection of the tension of the tether upon a line perpendicular to the vertical axis is constant [7].

Equilibria in a constant force field

General position solutions are (up to reflection across the Ox-axis and translation) given by

$$z = c_1 \cosh\left(\frac{x}{c_1}\right).$$

The, assumed positive, coefficient c_1 is simultaneously proportional to the dilation coefficient, the (imaginary) period and the horizontal tension component [2].

Although the problem of the hanging chain in a constant force field is regarded as a classical isoperimetric problem, the type of equilibrium solution (hyperbolic cosine) does not really depend upon the given length of the tether. The length of the tether does influence the type of solution in the case which we next consider.

Equilibria in a linear parallel force field

An exhaustive classification of tether equilibria in a linear parallel force field was given in [2]. Solutions are (up to reflection across and translation along the Ox-axis, as well as, a reflection across the Oz-axis) given by doubly periodic functions:

$$z = \begin{cases} z_* = \sqrt{\dfrac{c_2}{2}\left(\mathcal{R}_\beta\left(\dfrac{x}{\sqrt{2c_2}}\right) + \mathcal{R}_\beta^{-1}\left(\dfrac{x}{\sqrt{2c_2}}\right) + \beta + \beta^{-1}\right)} & : \ \beta + \beta^{-1} \in \mathbb{R}\backslash\{-2, 2\}, \\ \dfrac{c_2\left(\beta - \beta^{-1}\right)}{2\, z_*} & : \ \beta > 1, \end{cases}$$

separated into classes and subclasses by periodic function:

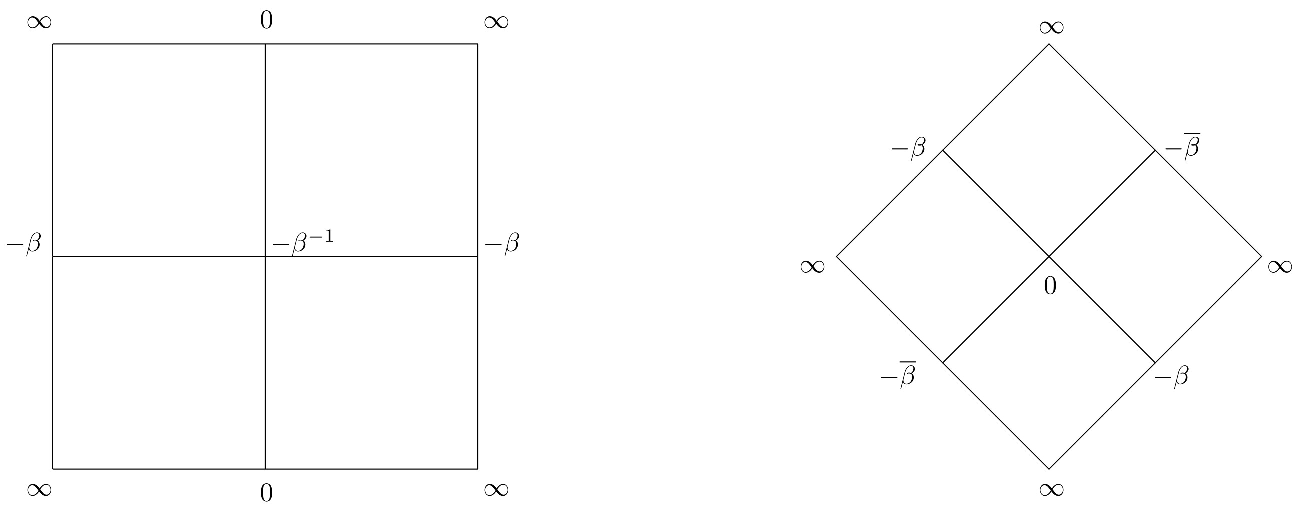
$$z = \sqrt{2\, c_2}\ \sec\left(\sqrt{\frac{2}{c_2}}\, x\right),$$

vertical (parallel to Oz) and horizontal-axial (along Ox) solutions. Here, the coefficient c_2 is assumed positive, whereas R_β is a second order elliptic function with a double pole at zero, whose values at the points where its derivative vanishes are 0, β and β^{-1}. The function R_β is an essential elliptic function [3], which is determined by a single parameter β, and differs by an additive constant from a Weierstrass elliptic function. The period parallelogram for R_β is a rectangle when β, excluding the two special values ± 1, is nonvanishing real, and is a rhombus when β is nonreal complex lying on the unit circle, centered at the origin. The period parallelogram is a square in two *central cases* represented in Fig. 2.

Three families of solutions in a linear parallel force field

The equilibria of a tether, with positive tension, in a field of linear parallel repelling force were determined in [8]. Three representatives of a family of equilibria passing through the points $(-1, 1)$ and $(1, 1)$ are displayed in Fig. 3. These solutions are bounded [1].

Fig. 4 exhibits (unbounded) equilibrium forms with vertical asymptotes at $x = 0$ and $x = 1$. Two (dual) interpretations might be given. The one-sided solutions (above the Ox-axis) might be viewed as equilibria of a tether, with positive tension, in a field of linear parallel attracting (to the Ox-axis) force. Alternatively, they might be viewed as solutions in a linear parallel repelling (from the Ox-axis) force, but the tension of the tether is now being assumed negative. From mechanical view point, negative tension might be interpreted as compression. One must stress, here, the duality between changing the sign of the tension and reversing the the direction of the force. The existence of a pair of dual to each other solutions guarantees the existence of



$\beta = \sqrt{2}$ $\beta = \sqrt{-1}$

Fig. 2: The rectangle and the rhombus as period parallelograms (with the square being a special case for both)

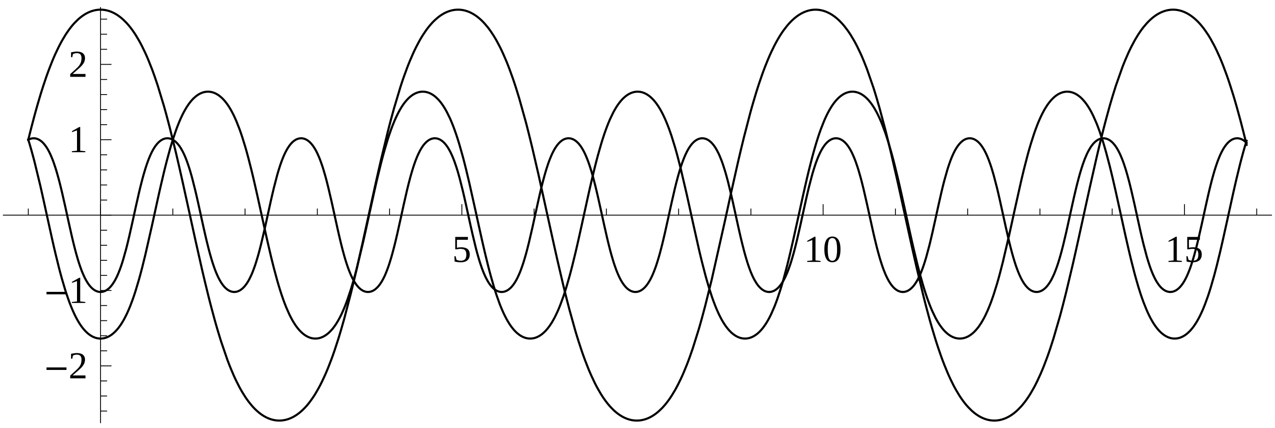


Fig. 3: Bounded equilibria passing through the points $(-1, 1)$ and $(1, 1)$

a fractional linear transformation between them. This transformation is invertible (and includes an inversion) if $\beta > 1$ and is not if $\beta = 1$. Non degenerate transformations may exist in the absence of dual solutions, namely, between unbounded solutions without extremal points [1]. These solutions are not dual to bounded solutions and might be viewed as solutions corresponding to imaginary tension, whereas the linear fractional transformation (including an inversion) between a pair of positive-imaginary and negative-imaginary tension solutions is equivalent here to a reflection along a vertical (parallel to the Oz-axis) [2]. The latter solutions (if analytically continued) do necessarily intersect the Ox-axis.

Fig. 5 exhibits equilibrium forms with the same nonzero magnitude of the horizontal component of the tension, corresponding to four values of β: $\sqrt{-1}$, 1, $\sqrt{2}$, $\left(3 + \sqrt{5}\right)/2$. The dotted line corresponds to the central value $\beta = \sqrt{-1}$. The three lines above it correspond, respectively, to the next three (increasing) values of β, with $\beta = 1$ corresponding to the special unbounded solution $z = \csc(2x)$ whose dual is the horizontal axis Ox itself. The bounded solution dual to the unbounded solution for the central value $\beta = \sqrt{2}$ is seen to itersect the solution without extremal points, corresponding to the same central value of β, at the Ox-axis. The bounded solution dual to the unbounded solution for the value $\beta = \left(3 + \sqrt{5}\right)/2$ attains a maximal value 1/2, and itersects the Ox-axis to the right of the intersection of the solution without extremal points, corresponding to that value of β, with the Ox-axis.

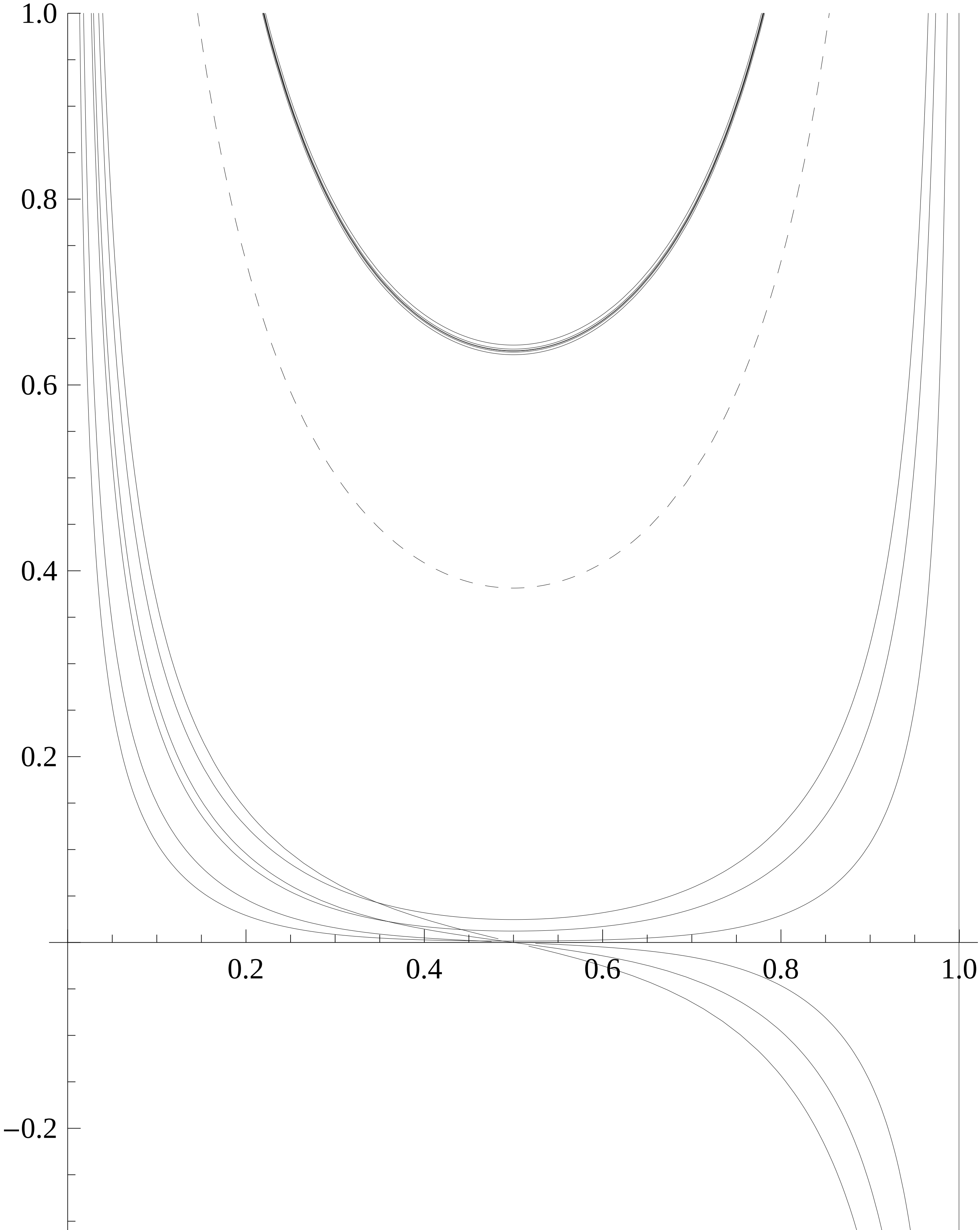


Fig. 4: Unbounded equilibria forms sharing two vertical asymptotes at $x = 0$ and $x = 1$

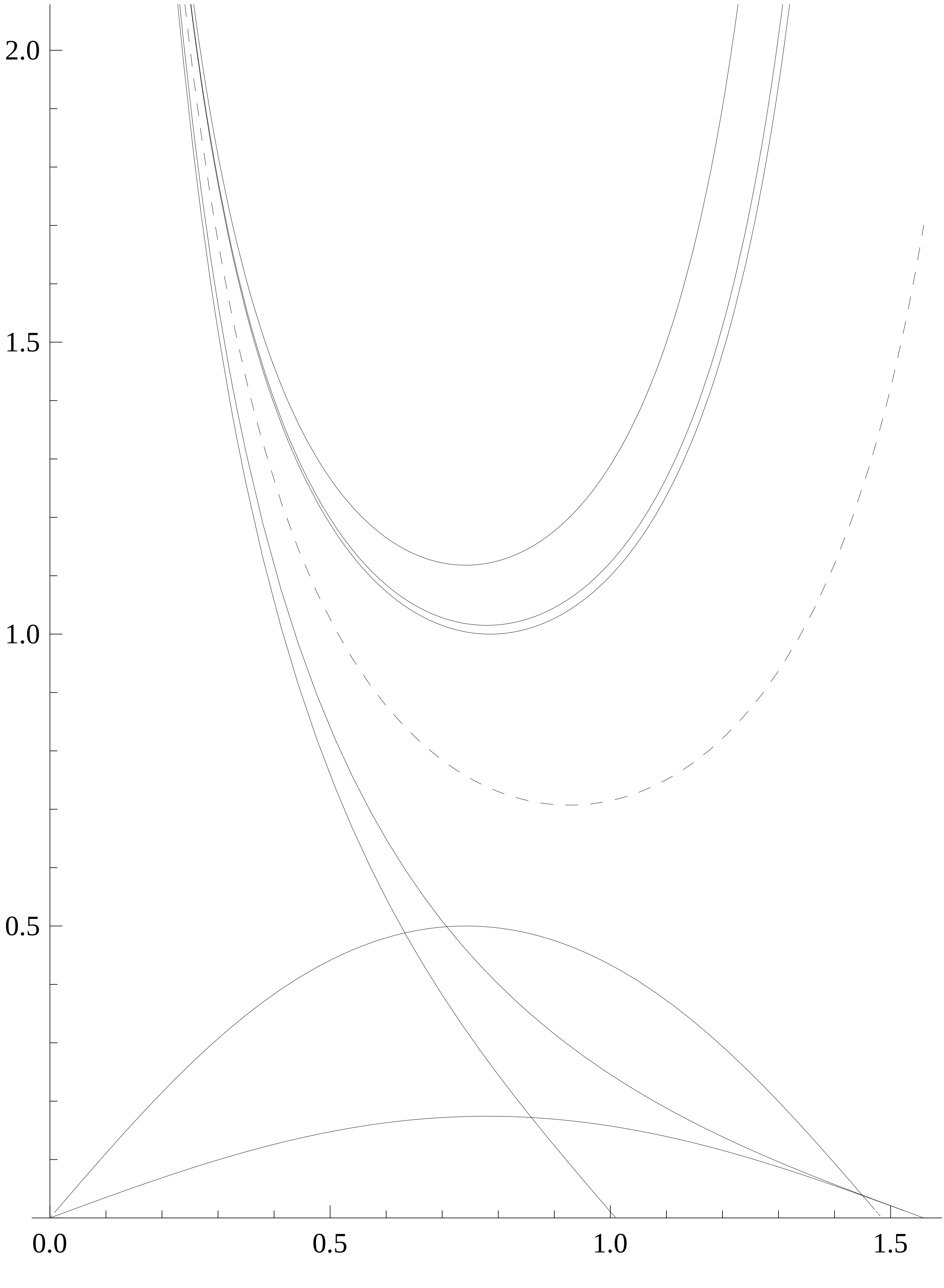


Fig. 5: Equilibrium forms sharing the same nonzero magnitude of the horizontal component of tension

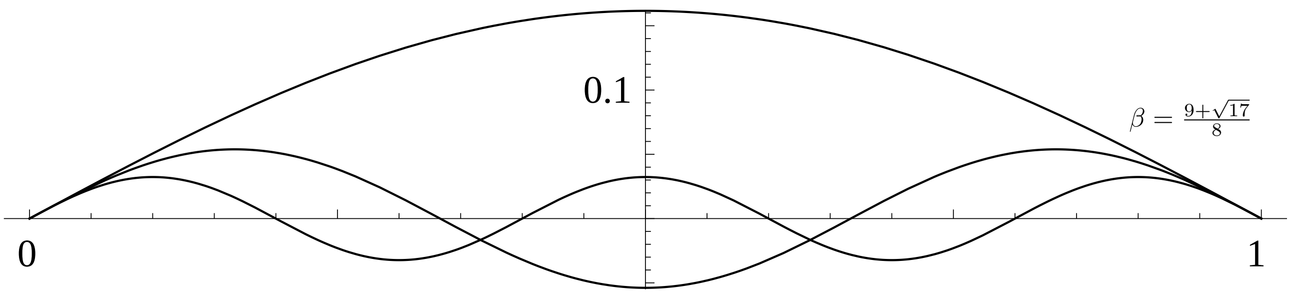


Fig. 6: Three equilibrium forms for a tether with a fixed length

For brevity, we have referred to the equilibria of a tether whose endpoints are fixed to an axis, the repelling force from which is proportional to the distance from it, as Appell equilibria. Appell had described a countably infinite family of homothetic, to each other, solutions, which might be chracterized by their number of half-waves: $1, 2, \ldots$ as in Fig. 6, where solutions with 1, 3 and 5 half-waves are represented. Pojaritsky [12] indicated that only the solution with minimal number of half-waves, i.e. the solution with a single half-wave, was stable. Implicitly, relying upon homothety, he (rightfully) claimed that other solutions were obviously unstable. Indeed, one might observe here that the potential of the tether (whose ends are fixed at $x = 0$ and $x = 1$) is inversely proportional to the number of half-waves, and, thus, an extremal value for the potential is attained at the single half-wave solution. Thereby, proving that the indicated family exhausts all tether equilibria for a given length constitutes a proof of uniquness of that stable solution, which we opt to express as:

$$z(x) = \frac{(\beta-1)M(\beta)}{\pi\beta} \ \operatorname{sn}\left(\frac{\pi(\beta+1)}{2M(\beta)} \ x, \ \frac{\beta-1}{\beta+1}\right), \ \beta > 1,$$

where $\operatorname{sn}(\cdot, k)$ is the Jacobi elliptic sine function with an elliptic modulus k. Denoting the length of the tether by l, and exploiting formula (2), we express it as a function of the parameter β:

$$l = l(\beta) = \frac{N(\beta^2)}{\beta}.$$

Since l is thereby seen as a strictly monotonous function of β, the latter parameter might be regarded as a strictly positive measure of the length. One might also note that the graph of the function z on the (closed) interval $[0, 1]$ appoaches the interval itself as β approaches 1, in agreement with the limiting length formula $l(1) = 1$.

A geometric approach to investigating the stability of a tether, whose end points are not restricted to being fixed to the axis, where the force vanishes, is described in [4].

Conclusion

In a linear parallel force field, the forms of equilibria are representable via doubly periodic (elliptic) functions dependent upon a parameter (β), and the constant (horizontal) component of tension might be interpreted (with proper choice of units) as a homothethy coefficient (c_2) for a corresponding lattice (in the complex plane). Such a lattice is invariant under multiplication by -1 and is rotated by a right angle if multiplied by $\sqrt{-1}$. With this rotation, an elliptic function, initially determined by a given parameter β, transforms into an elliptic function determined by the parameter $-\beta$. Allowing c_2 to being either real or imaginary, one might attain all solutions under the (seemingly restrictive) assumption that β is either positive exceeding 1 or a unit in upper right quadrant of the complex plane. Two special values must be pointed out. The value $\beta = 1$ corresponds to a special (trigonometric) unbouded equilibrium solution and to a horizontal-axial (coinciding with the Ox-axis) equilibrium solution with an undetermined tension. The central value $\beta = \sqrt{-1}$ corresponds to an unconditional extremal of the tether equilibrium problem in a linear parallel force field.

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