Torque free motion of a rigid body: from Feynman wobbling plate to Dzhanibekov flipping wingnut

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Le Problème de la rotation d'un corps solide quelconque,
qui n'est sollicité par aucune force accélératrice,
est susceptible d'être résolu par des formules nouvelles si élégantes et si parfaites,
que je ne peux m'empêcher de les communiquer à votre illustre Académie.[1]

Carl Gustav Jacob Jacobi

Abstract

The fundamental problem of torque free rigid body motion has traditionally bewildered researchers throughout the globe. A widely acclaimed geometric description of such motion via a rolling (without slipping) Poinsôt ellipsoid turns out being elusive in "critical" cases, evidenced by Dzhanibekov and demonstrated by Burke. As is the case with the simple pendulum "standing" in its unstable equilibrium position which "separates" two rotary motions (in two distinct directions): "clockwise" and "counterclockwise", a free rigid body spinning about its middle axis of inertia might "flip" in two distinct "oppositely oriented" ways! With the point at (complex) infinity being added to the time domain, of the critical solution, the uniqueness (which, otherwise, holds for a solution restricted to a bounded time domain) is violated. And, as was the case with the pendulum where two motion regimens (oscillatory and rotary) separated by unstable equilibrium must be distinguished, two motion regimens for a freely moving rigid body separated by "permanent" rotation about the middle axis must also be distinguished. Namely, a rotation might occur about either the "minor" or the "major" axis but it never occurs (simultaneously) about both. An analytic unifying solution of free rigid body motion, explicitly expressed via time dependent transition matrices, requires a (complete) determination of the group of its "preserving" fractional transformations. We shall discover that achieving an exact and computationally robust solution requires the construction of a fourth axis (along with body's three main axes of inertia), which we call Galois critical axis. It (and only it) rotates uniformly and permanently about the (fixed) angular momentum, even as the middle axis "reverses" its direction (to either match or oppose the direction of the angular momentum) during the critical motion which must (from now on) necessarily "augment" the permanent rotation about the middle axis, routinely characterized as "unstable" (mistakenly) suggesting that no other solutions emerge unless perturbations (however small) ensue. Moreover, dual critical solutions sharing one and the same (invariant) Galois critical axis must be "analytically continued at (complex) infinity" before we declare the motion completely determined and the problem entirely settled!

1 A prelude: Feynman's revelation of a source of inspiration

Richard Feynman told us in [11]:

"So I got this new attitude. Now that I am burned out and I'll never accomplish anything, I've got this nice position at the university teaching classes which I rather enjoy, and just like I read the Arabian Nights for pleasure, I'm going to play with physics, whenever I want to, without worrying about any importance whatsoever.

Within a week I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.

I had nothing to do, so I start to figure out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate two to one. It came out of a complicated

[1]The quote is taken from a letter [16, Extrait d'une lettre adressée à l'académie des sciences de Paris], read on July 30, 1849 at the Paris Science Academy:
The problem of the rotation of any solid body, which is not solicited by any accelerating force, is amenable to resolution by novel formulas so elegant and so perfect, that I can not help but communicate them to your glorious Academy.

equation! Then I thought, "Is there some way I can see in a more fundamental way, by looking at the forces or the dynamics, why it's two to one?" I don't remember how I did it, but I ultimately worked out what the motion of the mass particles is, and how all the accelerations balance to make it come out two to one.

It was effortless. It was easy to play with these things. It was like uncorking a bottle: Everything flowed out effortlessly. I almost tried to resist it! There was no importance to what I was doing, but ultimately there was. The diagrams and the whole business that I got the Nobel Prize for came from that piddling around with the wobbling plate."

Perhaps, this little story is even more amusing (and more telling) than Feynman thought, as he (inadvertently) reveals that he did not learn (or know) a (correct) solution to his problem, given in [19], by James Clerk Maxwell who had (incidentally!) cautioned that:

> "The theory of the rotation of a rigid system is strictly deduced from the elementary laws of motion, but the complexity of the motion of the particles of a body freely rotating renders the subject so intricate, that it has never been thoroughly understood by any but the most expert mathematicians. Many who have mastered the lunar theory have come to erroneous conclusions on this subject; and even Newton has chosen to deduce the disturbance of the earth's axis from his theory of the motion of the nodes of a free orbit, rather than attack the problem of the rotation of a solid body."

2 Preliminaries on the torque free motion of a rigid body

Let (throughout this article) A, B and C denote the values of the (three) principal moments of inertia of a rigid body, where B is (the value of) the middle moment. Whenever an axially symmetric rigid body is discussed, we shall assume that $A = B$. Thereby, while (always) imposing a lexicographic ordering upon the moments A, B and C, we (must) permit both ascending and descending orderings.

Let $\boldsymbol{m}$ and $\boldsymbol{w}$ denote the angular momentum and the angular velocity (pseudo)vectors, respectively.[2] The angular momentum of a freely rotating rigid body is conserved.[3] Let h denote twice the (conserved) kinetic energy, that is, $h := \boldsymbol{w} \cdot \boldsymbol{m}$, where the dot between vectors is the symbol of the scalar product operation. Let, from now on, p, q and r denote the (orthogonal) projections of the angular velocity $\boldsymbol{w}$ onto the principal (directed) axes of inertia with corresponding moments A, B and C. The directions of the corresponding unit vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, of the principal axes of inertia (upon which $\boldsymbol{w}$ was projected), might be so chosen so as to ensure right-handedness (whether A, B and C are ascendingly or descendingly ordered).[4] For brevity, we shall refer to the axis with the smallest (middle or largest) moment of inertia as the "minor" ("middle" or "major") axis.[5]

Put

$$a := h - \frac{m^2}{A}, \; b := h - \frac{m^2}{B}, \; c := h - \frac{m^2}{C},$$

and observe that the "initial" conditions

$$\begin{pmatrix} 1 & 1 & 1 \\ A & B & C \\ A^2 & B^2 & C^2 \end{pmatrix} \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} = \begin{pmatrix} w^2 \\ h \\ m^2 \end{pmatrix} \tag{1}$$

imply the identity

$$V \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} = \begin{pmatrix} BC(C-B)\left(w^2 - h^2/m^2 + bc/m^2\right) \\ CA(A-C)\left(w^2 - h^2/m^2 + ca/m^2\right) \\ AB(B-A)\left(w^2 - h^2/m^2 + ab/m^2\right) \end{pmatrix}, \tag{2}$$

where V is the determinant of the (Vandermonde) matrix on the left-hand side of (1), that is,

$$V := (A-B)(B-C)(C-A) = \frac{A^2B^2C^2(a-b)(b-c)(c-a)}{m^6}.$$

[2] Here, and throughout this article, we shall use the boldfaced letters to denote vectors, whereas their corresponding magnitudes are to be designated with the same letters in non-boldface.

[3] We are assuming that the motion is occurring with respect to an "inertial" frame, which we might refer to as the "absolute space", so the said preservation means that the vector $\boldsymbol{m}$ is fixed in the said absolute space. We might further assume that the origin of a "fixed" coordinate system in such space coincides with body center of mass, thereby constructing a (so-called) "König coordinate system". For brevity, we adopt the expression "free motion" instead of "torque free motion".

[4] Of course, no uniqueness of such choice is claimed. After all, the directions of any principal axes pair might be (simultaneously) reversed without altering the "right-handedness" of the ordered triple $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

[5] Note that the labels of the axes "minor" and "major" are better swapped if applied to describe body geometric proportions (contrary to our suggestion).

Here, we might imagine an action of the alternating three-element group A_3 upon the ordered triple of pairs (A, p^2), (B, q^2), (C, r^2) cyclically permuting it. Let τ denote the (non-trivial) element of A_3, determined by mapping (A, p^2) to (B, q^2), and identify A_3 with the additive group $\mathbb{Z}_3$ of integers modulo (the prime) 3, via mapping $\tau \in A_3$ to $1 \in \mathbb{Z}_3$. So we might, in particular, subject the first row of identity (2) to the action of the consecutive integers 0, 1 and 2 (viewed as elements of $\mathbb{Z}_3$), thereby obtaining successively all (three) rows of identity (2).

Assume, temporarily, that $b \neq 0$, so that, in particular, the tensor of inertia is not spherical. There are two general cases here, namely, the case $b < 0$ for which we impose the ordering $A \leq B < C$, and the case $b > 0$ for which we impose the ordering $A \geq B > C$. Either case, we have ensured that $ab > 0$.

Observe that identity (2) might be rewritten as

$$w^2 - \frac{h^2}{m^2} = \alpha^2 p^2 - \frac{bc}{m^2} = -\beta^2 q^2 - \frac{ca}{m^2} = \gamma^2 r^2 - \frac{ab}{m^2}, \tag{3}$$

$$\alpha := \sqrt{\frac{(A-B)(A-C)}{BC}},\ \beta := \sqrt{\frac{(A-B)(B-C)}{CA}},\ \gamma := \sqrt{\frac{(A-C)(B-C)}{AB}},$$

yielding the inequalities

$$0 \leq \sqrt{h^2 - ab} \leq h \leq \sqrt{h^2 - bc} \leq mw \leq \sqrt{h^2 - ca}. \tag{4}$$

Therefore, the angular speed w is constant (coinciding with h/m) if c vanishes. This is the case of "stable" permanent rotation about either the minor axis (if $b > 0$) or the major axis (if $b < 0$). Note that inequalities (4) might be transformed to inequalities, involving the third component r of the angular velocity, which we might assume to be positive, as it never vanishes, unless b does:

$$0 \leq \frac{\sqrt{ab}}{m\gamma} \leq \sqrt{\frac{Bb}{(B-C)C}} \leq r \leq \sqrt{\frac{Aa}{(A-C)C}}.$$ [6]

We must emphasize that the angular speed is not obliged to remain constant for vanishing b. In this critical case, which we shall fully explore, the bounds on the angular speed w might be rewritten as:

$$\frac{h}{m} \leq w \leq \frac{\delta h}{m},\ \delta := \sqrt{1 + \beta^2} = \sqrt{\frac{B(C + A - B)}{CA}}.$$

The preservation of the angular momentum $\boldsymbol{m}$ implies the (so-called) Euler equations of free motion of a rigid body

$$\frac{\partial \boldsymbol{m}}{\partial t} = \boldsymbol{m} \times \boldsymbol{w}, \tag{5}$$ [7]

which, combined with differentiating identity (3), imply the identity

$$\frac{dw^2}{dt} = -\frac{2Vpqr}{ABC}.$$

Thereby, the (elliptic) function $y := w^2 - h^2/m^2$ satisfies the differential equation

$$\dot{y}^2 = -4\left(y + \frac{bc}{m^2}\right)\left(y + \frac{ca}{m^2}\right)\left(y + \frac{ab}{m^2}\right). \tag{6}$$

Each of the (three) functions p^2, q^2 and r^2 coincide, up to a multiplicative constant, with an essential elliptic function $\mathcal{R}$, as defined in [3], which argument is shifted and dilated. In particular, we might arrange for the solution $y = y(t)$,[8] to the differential equation (6), to satisfy

$$y(t) + \frac{ca}{m^2} = -\beta^2\, q(t)^2 = \frac{(A-B)\,c}{AB}\ \mathrm{sn}\left(\sqrt{\frac{(B-C)\,a}{BC}}\,t,\ \frac{1}{k_2}\right)^2 = \frac{(C-B)\,a}{BC}\ \mathrm{sn}\left(\sqrt{\frac{(B-A)\,c}{AB}}\,t,\ k_2\right)^2 =$$

$$= -l_2^2\, \mathcal{S}\left(l_2\, t,\ k_2\right)^2,\ l_2 := \sqrt{\frac{\beta\sqrt{-ca}}{B}},\ k_2 := \sqrt{\frac{(B-C)Aa}{(B-A)Cc}}.$$

[6] Here, we might readily infer the constancy of r for an axially symmetric rigid body since then $A = B$. The angular speed $w = \sqrt{h^2 - bc}/m$ turns out being constant, as well.

[7] The partial time derivative is meant to emphasize differentiation with respect to body (moving) frame. The "total" time derivative $\dot{\boldsymbol{m}} := d\boldsymbol{m}/dt$ must vanish since $\boldsymbol{m}$ is constant in the absolute space.

[8] The argument t might be translated, if necessary.

where $\operatorname{sn}(\cdot,\, k)$ is the Jacobi elliptic sine function (with an elliptic modulus k), and the function $\mathcal{S}(\cdot,\, k)$ is the alternative elliptic function, as defined in [3].[9] Consequently,

$$q(t) = \frac{l_2\, \mathcal{S}\,(l_2\, t,\, k_2)}{\beta},\ r(t) = \frac{l_3\, \mathcal{S}\left(\sqrt{-1}\, l_3\, (t + T_1),\, k_3\right)}{\gamma},\ l_3 := \sqrt{\frac{\gamma\sqrt{ab}}{C}},\ k_3 := \sqrt{\frac{(C-A)Bb}{(C-B)Aa}},$$

$$p(t) = \frac{l_1\, \mathcal{S}\left(\sqrt{-1}\, l_1\, (t + \sigma T_3),\, k_1\right)}{\alpha},\ l_1 := \sqrt{\frac{\alpha\sqrt{bc}}{A}},\ k_1 := -\sqrt{\frac{(A-B)Cc}{(A-C)Bb}},\ \sigma := \operatorname{sgn}(A-C) = \operatorname{sgn} b,$$

where the function $\operatorname{sgn}(\cdot)$ is the "sign function", which value is 1 for positive arguments and -1 for negative arguments.[10] The values l_2, k_2, l_3 and k_3 are positive, whereas the values l_1 and k_1 are not real, so we must specify the branches of the square root to be chosen upon evaluating them. These branches might be determined via the (coupled) identities

$$l_1 l_2 l_3 = \frac{\sqrt{\sqrt{-1}\, abcV}}{ABC},\ k_1 k_2 k_3 = -\sqrt{-1},^{11}$$

which we impose. The alternating group A_3 acts upon the squares of the elliptic moduli k_1^2, k_2^2 and k_3^2, via the linear transformation $x \mapsto 1 - 1/x$,[12] as discussed in [4]. The quarter periods T_1, T_2 and T_3 are calculated as

$$T_1 = T_3 - T_2,\ T_2 = \frac{\sqrt{-ABC}\,\pi}{2\, M\left(\sqrt{(B-A)\, Cc},\, \sqrt{(B-C)\, Aa}\right)},\ T_3 = \frac{\sqrt{ABC}\,\pi}{2\, M\left(\sqrt{(B-C)\, Aa}\,,\, \sqrt{(A-C)\, Bb}\right)}, \tag{7}$$

where $M(x, y)$ is arithmetic-geometric mean of x and y.[13]

Define a mutually orthogonal unit vector pair:

$$\boldsymbol{u} := \frac{\boldsymbol{m}}{m},\ \boldsymbol{v} := \frac{m\boldsymbol{w} - h\boldsymbol{u}}{m\sqrt{y}}, \tag{8}$$

and verify that

$$\frac{\partial \boldsymbol{u}}{\partial t} = \sqrt{y}\, \boldsymbol{u} \times \boldsymbol{v},\ \frac{\partial \boldsymbol{v}}{\partial t} = \frac{abc\, \boldsymbol{u} \times \boldsymbol{v}}{m^3 y},\ \frac{\partial (\boldsymbol{u} \times \boldsymbol{v})}{\partial t} = -\sqrt{y}\, \boldsymbol{u} - \frac{abc\, \boldsymbol{v}}{m^3 y}.$$

Set

$$\boldsymbol{u}_\pm := s_\pm\, \boldsymbol{v} \pm s_\mp\, \boldsymbol{u} \times \boldsymbol{v},^{14}\ s_\pm := \sin\psi_\pm,\ \psi_\pm := \frac{\pi}{4} \pm \psi,\ \psi = \psi(t) := \frac{ht}{m} + \frac{abc}{m^3}\int_0^t \frac{dt}{y},^{15} \tag{9}$$

so that the constructed unit vector pair $\boldsymbol{u}_\pm$ satisfies

$$\boldsymbol{u}_- \cdot \boldsymbol{u}_+ = 0,\ \boldsymbol{u}_- \times \boldsymbol{u}_+ = \boldsymbol{u},\ \boldsymbol{w} \times \boldsymbol{u}_\pm = s_\pm h/m\, \boldsymbol{u} \times \boldsymbol{v} \pm s_\mp \left(\sqrt{y}\, \boldsymbol{u} - h/m\, \boldsymbol{v}\right),$$

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[9] The Jacobi elliptic sine function satisfies the identity $\operatorname{sn}(\cdot,\, k) = \operatorname{sn}(\cdot,\, -k)$, whereas the alternative elliptic function satisfies the identity $\mathcal{S}(\cdot,\, k) = \mathcal{S}(\cdot,\, 1/k)$. Both functions are odd with respect to the first argument. They are interrelated via the identity

$$\mathcal{S}(t,\, k) = \sqrt{k}\, \operatorname{sn}\left(\frac{t}{\sqrt{k}},\, k\right).$$

Most importantly, the elliptic functions $\mathcal{S}$ is the single "canonical" function via which the three functions p, q and r are expressed. Its square was already shown, in [3], to "naturally" determine the motion of a simple pendulum (in both oscillatory and rotary regimens).

[10] The latter equality (involving σ) is ensured since, thus far, the case with vanishing b has been excluded. Flipping the sign of σ would carry us from a right-handed to a left-handed orientation of the ordered triple $\{\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{i}\}$.

[11] The issue of determining the negative sign before the imaginary unit $\sqrt{-1}$ in the latter expression is most intricate. Meanwile, we might suggest an inspiring story, concerning Anna Johnson Pell Wheeler gorgeous formula for determining the signs of "an incomplete polynomial remainder sequence", told in [5]. We shall merely point out that Anna Johnson thesis (in 1904) titled "The extension of the Galois theory to linear differential equations" seems relevant here, and admit that our (negative) sign is entangled with the (imposed) ordering of the moments of inertia, the signs of (the imaginaru value) $l_1^2 l_2^2 l_3^2$, the (imaginary) quarter period T_2 and σ, as well as, the "orientation" of the elliptic function $\mathcal{S}$. Further clarification requires another (full length) article!

[12] Such transformation maps k_2^2 to k_3^2.

[13] The unsurpassably efficient calculation of complete elliptic integrals via the arithmetic-geometric mean was discovered (but not published!) by Gauss, as told in [2].

[14] Either the upper or the lower sign is presumed to be consistently chosen, whenver two sign choices arise.

[15] Few are led to believe that denoting the upper bound of an integral with the same letter used for the "dummy" variable is an "abuse" of notation, whereas it is not!

implying that the (ordered) triple $\{\boldsymbol{u}_-, \boldsymbol{u}_+, \boldsymbol{u}\}$ constitutes a right-handed orthonormal basis, which (furthermore) is fixed with respect to the absolute space, as we have ensured that the total time derivative of either vector $\boldsymbol{u}_-$ or $\boldsymbol{u}_+$ vanishes, as was already the case for $\boldsymbol{u}$.[16] Viewing the coordinates, of each vector of this triple, with respect to the (moving) basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, as the three components of a corresponding column of an orthogonal matrix Q, then the components of a row of Q are the coordinates, of a corresponding vector from the latter (moving) basis, with respect to the inertial basis $\{\boldsymbol{u}_-, \boldsymbol{u}_+, \boldsymbol{u}\}$. We must reemphasize that our construction of the (transition) matrix Q has relied on the assumption of the linear independence of $\boldsymbol{m}$ and $\boldsymbol{w}$, so that the complete settlement of the free rigid body motion problem requires the inclusion of the special (initially excluded) cases.

3 Free motion of an axially symmetric rigid body

The erroneously declared (by Feynman) spin to wobble ratio (2:1) was corrected in 1989 (after Feynman's death) by Benjamin Chao in [9]:[17]

"A torque free plate wobbles twice as fast as it spins when the wobble angle is slight. The ratio of spin to wobble rates is 1:2 not 2:1!"

Vladimir Arnold observes, in [6], that a freely rotating axially symmetric rigid body displays a (constant) precession which magnitude is m/B, where m is the magnitude of the (constant) angular momentum.

Hanspeter Schaub & John Junkins go on, in [26], to calculate the (constant) spin rate (for such a body) from which the spin to wobble ratio might be deduced as

$$B \cos\theta : C, \tag{10}$$

where θ is the (constant) nutation angle, that is, the angle between the angular momentum and the (directed) axis of symmetry. Although they did not tell us how to measure the angle θ (in terms of energy and momentum), we might readily observe a limiting case, where the angle θ vanishes (so that the axis of symmetry is aligned along the angular momentum) and the spin to wobble ratio is then $B : C$, consistently with Chao's observations.

Here, we shall disclose an elementary derivation for $\cos\theta$,[18] so note that, along with the constant scalars m and h, the magnitude w, that is, the angular speed (for an axially symmetric rigid body) is also preserved, as

$$h = B(p^2+q^2) + Cr^2,\ m^2 = B^2(p^2+q^2) + C^2r^2,\ w^2 = p^2+q^2+r^2 = \frac{(B+C)h - m^2}{BC}. \tag{11}$$

As mentioned earlier, the directions of the corresponding unit vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, of the principal axes of inertia (upon which $\boldsymbol{w}$ was projected), might be so chosen so as to ensure right-handedness of the (body) coordinate system and that the (constant) projection of $\boldsymbol{w}$ onto the directed axis of symmetry (being directed along $\boldsymbol{k}$), which we have already presumed to be r, is non-negative,[19] that is,

$$\boldsymbol{w}\cdot\boldsymbol{k} = r = \sqrt{\frac{Bb}{(B-C)C}}.$$ [20]

So we have

$$\boldsymbol{m} = B\,(p\,\boldsymbol{i} + q\,\boldsymbol{j}) + Cr\,\boldsymbol{k},\ \ \boldsymbol{w} = p\,\boldsymbol{i} + q\,\boldsymbol{j} + r\,\boldsymbol{k} = \frac{\boldsymbol{m}}{B} + \frac{(B-C)r\,\boldsymbol{k}}{B},$$

and the "relative" spin to wobble ratio is

$$\frac{(B-C)\,r}{m} = \frac{\sigma}{m}\sqrt{\frac{(B-C)Bb}{C}}. \tag{12}$$ [21]

[16] So we claim that $\partial\boldsymbol{u}_\pm/\partial t = \boldsymbol{u}_\pm \times \boldsymbol{w}$, as might readily be verified.

[17] Having investigated "the Chandler wobble phenomenon", Chao knew the correct ratio before he came across Feynman's error.

[18] Without even requiring the preliminaries of the preceding section.

[19] We have already disclaimed uniqueness in choosing body coordinate system (in footnote 4). Here, furthermore, (any) two principal axes, orthogonal to the axis of symmetry, remain principal if simultaneously (but arbitrarily) rotated about the said axis of symmetry.

[20] Actually, we deduced this formula earlier. See footnote 6.

[21] The nutation angle θ might thus be calculated as $\theta = \mathrm{Arccos}\,(Cr/m)$, so it can be chosen to (always) lie in the closed interval $[0, \pi/2]$.

Note again that the left hand side of equation (12) vanishes, for $B = C$, simultaneously with the expression inside the square root on its right hand side, so that an emerging ambiguity of σ (at $B = C$) need not necessarily be resolved. Now, the latter expression (12) is no longer consistent with the former expression (10). Note, however, that it is consistent with notion of the direct (for $B > C$) and retrograde (for $B < C$) precession, as discussed in [26]. There, the (body) spin about its axis of symmetry (with rate r) is distinguished from the "relative" spin (with rate $(B - C)\,r/B$). The later rate might, as well, be viewed as the rate of change of the third Euler angle, which we shall denote by ϕ, that is, the angle of "proper" rotation about the third Euler axis (which coincides, in our case, with body axis of symmetry). Therefore, the ratio given by (12) is the ratio $\dot{\phi} : \dot{\psi}$,[22] where ψ is the first Euler angle, that is, the precession angle.[23] For the "rod" (which moment of inertia about its symmetry axis vanishes, that is, $C = 0$), which Chao also mentions in [9], ratios (10) and (12) agree (with Br/m). He observes that the "rod" (upon assuming that $m = Cr$) "does not wobble". We must, nevertheless, emphasize that the case indicated by Chao is merely a limiting case, for which the wobble is not usually viewed as such (regardless of the relative magnitudes of B and C). In other words, for $m = Cr$, the motion might always be interpreted as "pure" spinning.[24] Yet, the "rod" does indeed represent a (most) special case (even for nonvanishing m) which must rightfully be singled out, as the dot product $\boldsymbol{m} \cdot \boldsymbol{k}$ vanishes, implying that the spinning and the wobbling do not "influence" each other. Note that in the limiting case with $m = Cr$ (as the nutation angle θ vanishes, so that the directions of $\boldsymbol{m}$ and $\boldsymbol{k}$ coincide with each other), we have $\dot{\psi} = r - \dot{\phi} = Cr/B$, so (in particular) $\dot{\psi} = 0$ if $C = 0$, and $\dot{\psi} = 2r$ if $C = 2B$.[25] For the totally symmetric case with $B = C$, we then have $\dot{\psi} = r$.[26] For another limiting case, where $m = Bq$, we have $\dot{\psi} = q$, no matter what C is. Once again, we are free to interpret the latter motion either as "pure" wobbling or "pure" spinning.

4 A generalized spin to wobble ratio

Recall that the angular velocity $\boldsymbol{w}$ in body rotating frame is (doubly) periodic,[27] which (real) quarter period T_3, was calculated in (7).

A formula for calculating the rate of precession $\dot{\psi}$, symmetric in the moments of inertia (A, B and C),

$$\dot{\psi} = \frac{1}{m}\left(h + \frac{abc}{m^2 y}\right),^{28} \tag{13}$$

was first presented at the PCA annual conference (chaired by Nikolay Vassiliev) on April 20, 2016 [1]. Thus, the generalized spin to wobble ratio might formally be defined and explicitly calculated as

$$\frac{\sigma\,\pi}{2\,\psi(T_3)}, \quad \psi(t) = \int_0^t \dot{\psi}\, dt. \tag{14}$$

[22] This ratio is $-1 : 2$ for "Feynman's wobbling plate". It differs in sign from the ratio $1 : 2$, which Chao had (correctly) calculated. These two seemingly contradictory ratios correspond to two distinct interpretations of precession. According to the first, the axis of symmetry is intrinsic to the body, that is, the axis itself moves with the body. Whereas, according to the second (which Feynman adopts), the axis of symmetry is detached from the body, so the rest of the body "spins" around it "independently" of its own movement. The first ratio, according to the first interpretation means that the spinning and the wobbling possess opposite directions. Adding up two opposed magnitudes -1 and 2 yields 1, which according to the second interpretation is the relative magnitude of the spinning which is codirected with the wobbling (which relative magnitude is still 2).

[23] The precession angle ψ ought not be confused with the second Euler angle, that is, the nutation angle θ, which Chao referred to as "the wobble angle". The nutation angle θ is also, quite frequently, referred to as "the pitch angle".

[24] We were made aware of two interpretations of precession since (unconsciously) adopting the second is inevitably followed by the (seemingly natural) additional assumption that the axis of symmetry "does not spin" (since its own spinning would not then influence the spinning of the rest of the body around it), thereby missing the alternative and important interpretation of the case $m = Cr$ as "pure" wobbling. No problems emerge, if the first interpretation is adopted, since the spinning of an axis (with vanishing thickness) might still be defined as long as the axis is "firmly" attached to the rest of the body. So, in fact, the first interpretation is preferable, although we must keep the second in mind (as Chao did), since it is rather commonly (and unconsciously!) assumed.

[25] This is the limiting case, which in terms adopted by Chao, is described as the case when "the rod does not wobble" and "the plate wobbles twice as fast as it spins".

[26] Chao had (wisely) avoided discussing this (spherically symmetric) case, for which $\dot{\phi} = 0$. Perhaps, he avoided (generally) stating that "the ratio of spin to wobble rates" is $B : C$ in order not to overburden the readers with the inevitable conclusion that the "sphere" spins as fast as it wobbles! Chao's "graciousness" (towards Feynman) somehow precluded a (total) clarification of Feynman's (not so insignificant) error, which was not in the least "a mere slip of memory"! Feynman did not finish deriving the "complicated equation", that is, he did not arrive at the said (simple) ratio $B : C$ which would have protected him from the "pretty obvious" recollection that the spinning of the plate went "faster than the wobbling".

[27] Thereby, the (scalar) function w is, as well, (doubly) periodic (in any reference frame).

[28] The search for this symmetric expression was inspired and guided by Galois. In fact, the search for the invariant axes and their construction is altogether due to Galois, who was undoubtedly able to carry out and surpass all that we have done!

The expression for calculating the precession angle ψ is, of course, the expression obtained on the right hand side of (9). Thus, $\psi(T_3)$ is a complete elliptic integral of the third type, which definition along with its (most efficient) calculation was exhaustively discussed in [2].[29] We might explicitly calculate it as

$$\psi(T_3) = \psi\left(A, B, C, h, m^2\right) := \frac{T_3}{m}\,(h + H)\,,^{30} \tag{15}$$

$$\frac{T_3}{m} = \frac{\pi}{2\,M\left(\sqrt{a(b-c)}, \sqrt{b(a-c)}\right)} = \frac{\pi}{2\sqrt{a(b-c)}\,M(k_3)} = \frac{\pi}{2\sqrt{b(a-c)}\,M(1/k_3)},\ k_3 = \sqrt{\frac{b\,(a-c)}{a\,(b-c)}},$$

$$H := \frac{abc}{T_3 m^2}\int_0^{T_3}\frac{dt}{y} = \int_{-bc/m^2}^{-ca/m^2}\frac{abc\,dy}{T_3 m^2 y\sqrt{-4(y+bc/m^2)(y+ca/m^2)(y+ab/m^2)}} = aH_1 = bH_2 = cH_3,$$

$$H_1 := \frac{2}{\pi}\,M\left(\frac{1}{k_3}\right)\int_0^1\frac{dx}{((b-a)\,x^2/b-1)\sqrt{(1-x^2)(1-k_1^2\,x^2)}} = N\left(k_3^2,\,0,\,\frac{a-c}{a+c},\,\frac{a-c}{a}\right),$$

$$H_2 := \frac{2}{\pi}\,M(k_3)\int_0^1\frac{dx}{((a-b)\,x^2/a-1)\sqrt{(1-x^2)(1-x^2/k_2^2)}} = N\left(\frac{1}{k_3^2},\,0,\,\frac{b-c}{b+c},\,\frac{b-c}{b}\right),$$

$$H_3 := \frac{2}{\pi}\,M(k_3)\int_1^{1/k_3}\frac{dx}{((a-c)\,x^2/a-1)\sqrt{(x^2-1)(1-k_3^2\,x^2)}} = N\left(k_3^2,\,0,\,\frac{a-c}{2\,a},\,\frac{a-c}{a}\right) =$$

$$= N\left(\frac{1}{k_3^2},\,0,\,\frac{b-c}{2\,b},\,\frac{b-c}{b}\right),^{31}$$

where $M(x)$ is the arithmetic-geometric mean of 1 and x, whereas the function $N(x,\zeta,\eta,\xi)$ is defined recursively via the relation

$$N\left(x_n,\,\zeta_n,\eta_n,\xi_n\right) = N\Big(x_{n+1} := \sigma(x_n,1),\,\zeta_{n+1} := \sigma(x_n,\zeta_n,\xi_n),\,\eta_{n+1} := \sigma(x_n,\eta_n,\xi_n),\,\xi_{n+1} := \sigma(x_n,\xi_n)\Big),$$

$$\sigma(x,\xi) := \sigma(x,\xi,\xi),\ \ \sigma(x,\eta,\xi) := \frac{(\sqrt{x}+\eta)\,(\sqrt{x}+\xi)}{2\,(\eta+\xi)\,\sqrt{x}},$$

and the value of this recursive function is the limit obtained from successively applying linear fractional transformations

$$L(x,\zeta_n,\eta_n,\xi_n) := \frac{(\eta_n-\xi_n)(x-\zeta_n)}{(\eta_n-\zeta_n)(x-\xi_n)},$$

either to (successive) corresponding first arguments x_n, thereby generating the sequence $\{L(x_n,\zeta_n,\eta_n,\xi_n)\}$, or to the (constant) value 1,[32] generating the sequence $\{L(1,\zeta_n,\eta_n,\xi_n)\}$. Both sequences converge quadratically to their common point, that is, the generalized arithmetic-geometric mean, as further clarified in [2].

Now if $A = B$ then $a = b$ and $2T_3 = \pi\sqrt{BC}/\sqrt{(B-C)b}$, as evident from (7). Thus, for an axially symmetric body, where $\dot\psi$ is constant,[33] the spin to wobble ratio, as given via formula (14), is reduced to formula (12), since then $\psi(T_3) = m\,T_3/B$.[34] The latter angle might also be regarded as a limiting case of the general formula (15), where the generalized arithmetic-geometric sequence converges in a single step! In fact, we then have

$$H_1 = H_2 = N\left(1,\,0,\,\frac{b-c}{b+c},\,\frac{b-c}{b}\right) = L\left(1,\,0,\,\frac{b-c}{b+c},\,\frac{b-c}{b}\right) = -1,$$

$$H_3 = N\left(1,\,0,\,\frac{b-c}{2\,b},\,\frac{b-c}{b}\right) = L\left(1,\,0,\,\frac{b-c}{2\,b},\,\frac{b-c}{b}\right) = -\frac{b}{c},$$

so that (either way) $H = -b$, consistently with our earlier calculations.

At a critical separating solution of a freely rotating axially symmetric rigid body, as b (strictly) vanishes, contrary to the (preliminary) assumption, the (constant) second summand of the identity for $\dot\psi$, given in (13), vanishes (as it coincides with $-b$) and the rate of precession $\dot\psi$ is constant at $h/m = m/B = q$.[35] Hence, the

[29] A relevant iterative procedure for calculating complete elliptic integrals of the third kind is given, as well, in [15, Appendix].

[30] Observe that the function $\psi(A,B,C,h,m^2)$ is homogeneous of degree 0 whether viewed as a function of the (principal) moments of inertia (for fixed energy and momentum), or as a function of h and m^2 (for fixed moments of inertia). This property might be expressed via the relation

$$\psi\left(\lambda A,\lambda B,\lambda C,\mu h,\mu m^2\right) = \psi\left(A,B,C,h,m^2\right),$$

where λ and μ might (unnecessarily!) be restricted to be positive.

[31] An alternative equivalent expression is given in [1].

[32] The sequence of first arguments $\{x_n\}$ converges quadratically to 1.

[33] The constancy of $\dot\psi$ might formally be derived from the general identity (13) by observing that the case $A = B$ would imply, by (11), that $m^2w^2 - h^2 = -bc$ and whence $\dot\psi = (h-b)\,/m = m/B$.

[34] Of course, formulas (14) and (12) still agree in the case of a spherically symmetric tensor, for which $T_3 = \infty$.

[35] Although, then $T_3 = \infty$ (mis)leading many authors to (routinely) ignore this crucial case!

afore discussed (at the end of the preceding section) free motion of an axially symmetric rigid body, where $m = Bq$, might also be viewed here as a special case of critical motion (with vanishing b) for which constant precession is displayed along with vanishing "relative" spin $\dot{\phi} \equiv 0$.[36] However, the nutation angle θ, whenever the moments of inertia are pairwise distinct (unlike the case of an axially symmetric rigid body), is no longer constant. And, moreover, it cannot be restricted from acquiring all(!) values in the range $[0, \pi]$ during the (critical) motion (with $b = 0$), so that $\cos\theta$ assumes both positive and negative values. A striking observation (in an orbiting space station) of a free motion in "proximity" to a critical solution, where a "quick" transition between two extreme values of θ occurs,[37] was made in 1985 (June 25$^{\text{th}}$) by the Soviet cosmonaut Vladimir Dzhanibekov.[38] A video demonstration [10], of this spectacular motion, got the attention of Terrence Tao, who shared his interpretation publicly (on Google+) [27]:

"The tennis racket theorem asserts that when rotating a rigid body with three distinct moments of inertia, the rotation around the axes with the largest or smallest moments of inertia is stable, but the rotation around the axis with the intermediate moment of inertia is unstable. Indeed, in the latter case the object will (when one looks just at the angular velocities) typically traverse periodically through the space of all states with the given angular momentum and energy, which is a closed curve known as a herpolhode that will pass close to both antipodes of the unstable equilibrium in an alternating fashion."

A (more accurate) explicit description of "the twisting tennis racket" phenomenon was given in [7]:

"The classical treatments of the dynamics of a tennis racket about its intermediate axis fail to describe a remarkable aspect of its motion which is revealed in the following experiment. Mark the faces of the racket so that they can be distinguished. Call one rough and the other smooth. Hold the racket horizontally by its handle with the smooth face up. Toss the racket into the air attempting to make it rotate about the intermediate axis (namely, the axis in the plane of the face which is perpendicular to the handle). After one rotation, catch the racket by the handle. The rough face will almost always be up! In other words, the racket typically makes a half-twist about its handle."

The authors go on (justly) stating that:

"The twisting phenomenon seems to be new. It is not mentioned in a recent article on the Eulerian wobble (Colley, 1987), in general texts on classical mechanics (Arnol'd, 1978; Goldstein, 1950; Landau and Lifschitz, 1976), or in specialized texts on rigid body motion (Klein and Sommerfeld, 1897-1910; Webster, 1920)."

"The experiment" appears to be due to William Burke, whose life was tragically abrupted in 1996, before Dzhanibekov's observation was made publicly accessible. Yet he brought to our attention a key observation "a half-twist" which is as crucially relevant in describing the motion of Dzhanibekov wingnut as it is relevant for describing the motion of the tennis racket. Recent articles, devoted to Dzhanibekov's observation, such as [22], have fallen short from delving into explicit mathematical reconstruction, leaving it for us to be carried out here and now![39]

5 Critical motion of a non-axially symmetric rigid body

Assume the critical case of motion ($b = 0$) of a rigid body which inertia moments A, B and C are pairwise distinct. With B being (as already stipulated) the value of the middle moment, the three values are (necessarily) lexicographically ordered. Relabeling, if necessary, we might and shall enforce (at will) an ascending ordering $A < B < C$. Put

$$\mathscr{B} := \sqrt{B(C-A)},\ \mathscr{C} := \sqrt{C(B-A)},\ \mathscr{A} := \sqrt{A(C-B)},$$

and observe the identities

$$\beta = \frac{\mathscr{C}\mathscr{A}}{CA} = \frac{(B-A)\mathscr{A}}{\mathscr{C}A} = \frac{\mathscr{C}(C-B)}{C\mathscr{A}}.$$

Observe, furthermore, that we might always determine an angle u, in the first quadrant,[40] satisfying the (simultaneous) identities $\cos u = \mathscr{C}/\mathscr{B}$ and $\sin u = \mathscr{A}/\mathscr{B}$.

[36] The case of a spherical tensor of inertia might also be viewed as a further subcase of this special case.

[37] The "quickness" of transition (between two extreme values of θ) is the reason for describing the motion of Dzhanibekov wingnut as "flipping".

[38] Vladimir Dzhanibekov had successfully led (June 6$^{\text{th}}$ - September 26$^{\text{th}}$, 1985) a most challenging space mission docking Soyuz T-13 with the "dead" space station Salyute 7, as told in [18, 25].

[39] We must, in particular, determine whether the "half-twist" of the racket occurs about its handle or about some other "nearby" axis!?

[40] We might "visualize" the said quadrant in a (planar) coordinate system spanned by the (ordered) vector pair $\boldsymbol{k}$ and $\boldsymbol{i}$.

The function $y = w^2 - m^2/B^2 = \beta^2\left(m^2/B^2 - q^2\right) = \gamma^2 r^2 = \alpha^2 p^2$ satisfies the differential equation

$$\dot{y}^2 = 4\,y^2\left(\frac{\beta^2 m^2}{B^2} - y\right), \tag{16}$$

which is a special case of the differential equation (6), with vanishing b, possessing a solution $y(t) = \left(\beta m/B\,\mathrm{sech}\,(\beta m t/B)\right)^2$. Thus, we might write

$$w(t) = \frac{m}{B}\sqrt{1 + \left(\beta\,\mathrm{sech}\left(\frac{\beta m t}{B}\right)\right)^2}.$$

One might, as well, verify that the (three) functions

$$p = p(t) = \frac{m\sin u}{A}\,\mathrm{sech}\left(\frac{\beta m t}{B}\right),\ q = q(t) = \frac{m}{B}\tanh\left(\frac{\beta m t}{B}\right),\ r = r(t) = \frac{m\cos u}{C}\,\mathrm{sech}\left(\frac{\beta m t}{B}\right)$$

satisfy the Euler equations (5) in our critical case (with $b = 0$).[41] Therefore, we have

$$\boldsymbol{w} = p\,\boldsymbol{i} + q\,\boldsymbol{j} + r\,\boldsymbol{k} = \frac{\boldsymbol{m} + (B-A)\,p\,\boldsymbol{i} + (B-C)\,r\,\boldsymbol{k}}{B} = \frac{\boldsymbol{m} + \beta\,m\,\mathrm{sech}(\beta m t/B)\,\boldsymbol{v}}{B},\ \ \boldsymbol{v} := \cos u\,\boldsymbol{i} - \sin u\,\boldsymbol{k},^{42}$$

$$\boldsymbol{m} = Ap\,\boldsymbol{i} + Bq\,\boldsymbol{j} + Cr\,\boldsymbol{k} = m\left(\tanh(\beta m t/B)\,\boldsymbol{j} + \mathrm{sech}(\beta m t/B)\,\boldsymbol{n}\right),\ \ \boldsymbol{n} := \sin u\,\boldsymbol{i} + \cos u\,\boldsymbol{k}.$$

So, in body frame, the trajectory of the "tip" of the angular momentum $\boldsymbol{m}$ is nothing but half a circle, of radius m, whereas the trajectory of the "tip" of the angular velocity $\boldsymbol{w}$ is nothing but half an ellipse, which semi-minor axis length is m/B. The length of its semi-major axis is $\delta m/B$. The semi-ellipse is, perhaps, better seen if we reexpress $\boldsymbol{w}$ as

$$\boldsymbol{w} = \frac{m}{B}\left(\tanh\left(\frac{\beta m t}{B}\right)\boldsymbol{j} + \mathrm{sech}\left(\frac{\beta m t}{B}\right)(\beta\,\boldsymbol{v} + \boldsymbol{n})\right).$$

Note that the unit vectors $\boldsymbol{v}$ and $\boldsymbol{n}$ are orthogonal to each other, as well as, they are orthogonal to the vector $\boldsymbol{j}$. We might reconstruct the vectors $\boldsymbol{k}$ and $\boldsymbol{i}$ back from the vectors $\boldsymbol{v}$ and $\boldsymbol{n}$ as

$$\boldsymbol{k} = \cos u\,\boldsymbol{n} - \sin u\,\boldsymbol{v},\ \ \boldsymbol{i} = \cos u\,\boldsymbol{v} + \sin u\,\boldsymbol{n}.$$

We shall name the axis spanned by the vector $\boldsymbol{v}$ *Galois critical axis*,[43] and we shall denote the square $(\cos u)^2$ by G and call it *Galois critical modulus*. The value G coincides with the square of the dot product of the vector $\boldsymbol{v}$ with the vector $\boldsymbol{i}$ (which lies along the minor axis).[44] It is bounded below by zero and above by one. The bounds are attained for an axially symmetric body with either $A = B < C$ or $A < B = C$,[45] respectively.

Most importantly, we must no longer insist either on the positivity or (even) the non-negativity of the third (in body frame) component r, of the angular velocity $\boldsymbol{w}$. Rather, we must not omit a dual solution $\boldsymbol{w}_*$, which satisfies

$$\boldsymbol{w}_*(t) = \boldsymbol{w}(t + 2T_2),$$

where T_2 is the (imaginary) quarter period, as calculated in (7), that is, $2\,T_2 = \sqrt{-1}\,\pi\,B/(\beta m)$. The dual solutions $\boldsymbol{w}$ and $\boldsymbol{w}_*$, while distinct for all real values of t, do coincide in their limits as t approaches infinity (via either the negative or the positive axis), that is,

$$\boldsymbol{w}_*(\pm\infty) = \boldsymbol{w}(\pm\infty) = \pm\frac{m}{B}\,\boldsymbol{j}.$$

[41] Evidently, the "tip" of the vector $\boldsymbol{w}$ traces a planar curve since the vector $A\cos u\,\boldsymbol{i} - C\sin u\,\boldsymbol{k}$ (which is constant in body's frame) is always (that is, for all t) orthogonal to $\boldsymbol{w}(t)$.

[42] The definition of the unit vector $\boldsymbol{v}$, in this special case, is consistent with its (general) definition, as given by (8)! Although, for $b = 0$, neither $\boldsymbol{w}$ nor $\boldsymbol{u}$ is fixed in body frame, the vector $\boldsymbol{v}$ is!

[43] It might still "inherit" the direction of v.

[44] So, in fact, the "half-twist" of the "tennis racket" does not exactly occur about its handle. It rather occurs about Galois critical axis! For a "flat" body, the moments of inertia satisfy the equality $A + B = C$, and so we must have $G = (2\beta/\delta^2)^2$. Dzhanibekov wingnut has moments of inertia proportional to 2, 7 and 8, as calculated in [22], so $\beta^2 = 5/16$ and $G = 20/21$. Its critical motion might be "modeled" by a "flat" body, which moments of inertia are proportional to 11, 21 and 32 (so that $G = 320/441$, whereas the value of β^2 remains unaltered).

[45] These two cases were united when we considered an axially symmetric body as $A = B \neq C$. Galois critical axis coincides, up to a sign, with the directed axis of symmetry of an axially symmetric body. More precisely, the directions coincide if the axis of symmetry further coincides with the minor axis and are opposite each to the other if the axis of symmetry turn out to be the major axis. Only for a body with a spherically symmetric tensor of inertia does Galois critical axis cease to be defined.

The said duality might be represented by assigning dual values (1 and -1) to σ in the expression for the angular velocity $\sigma p\, \boldsymbol{i} + q\, \boldsymbol{j} + \sigma r\, \boldsymbol{k}$.[46] So we might proceed with assuming a fixed sign for σ, while keeping in mind that both signs $\pm$ are possible.

Set the time unit so as $m/B = h/m = \sqrt{h/B} = 1$, and construct the vector pair $\boldsymbol{u}_\pm$, as in (9),

$$\boldsymbol{u}_\pm = s_\pm\, \boldsymbol{v} \pm s_\mp \left(\sigma \operatorname{sech}(\beta t)\, \boldsymbol{j} - \tanh(\beta t)\, \boldsymbol{n}\right), \; s_\pm = \sin\left(\frac{\pi}{4} \pm t\right).$$

Shifting the argument t (by subtracting $\pi/4$), we might alternatively set $s_- = \cos t$, $s_+ = \sin t$, and proceed to constructing the matrix Q as a transition matrix from observer's "inertial" coordinates to body "rotating" coordinates:

$$Q = Q(t) = Q(t, \beta, \sigma) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sigma \operatorname{sech}(\beta t) \sin t & \sigma \operatorname{sech}(\beta t) \cos t & \tanh(\beta t) \\ \tanh(\beta t) \sin t & -\tanh(\beta t) \cos t & \sigma \operatorname{sech}(\beta t) \end{pmatrix}$$

We shall, throughout this paper, imply that the coordinates of a row vector are given in observer's inertial frame. A row of the matrix Q provides the (three) coordinates of a corresponding vector, from the "rotating" orthogonal basis $\{\boldsymbol{v},\, \boldsymbol{j},\, \boldsymbol{n}\}$, with respect to the "inertial" basis $\{\boldsymbol{u}_-,\, \boldsymbol{u}_+,\, \boldsymbol{u}\}$,[47] whereas a column of Q provides the coordinates of a corresponding vector, from that inertial basis, with respect to body newly designated basis. Note, in particular, that the third column, viewed as a vector in body frame, coincides with the (unit) vector $\boldsymbol{u} = \boldsymbol{m}/m$, which (of course) is a fixed vector in our inertial frame, coinciding with the third basis vector $(0, 0, 1)$. We might also verify that

$$\dot{Q}\bar{Q} = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}, \; w_1 = \sigma\beta \operatorname{sech}(\beta t), \; w_2 = \tanh(\beta t), \; w_3 = \sigma \operatorname{sech}(\beta t).$$

In observer's frame, the angular velocity $\boldsymbol{w}$ has constant third coordinate, whereas the other two coordinates might be viewed as coordinates of a (planar) curve, known as the herpolhode:

$$\boldsymbol{w}(t) = \boldsymbol{w}(t, \beta, \sigma) = (\sigma\beta \operatorname{sech}(\beta t) \cos t,\, \sigma\beta \operatorname{sech}(\beta t) \sin t,\, 1)\,.$$ [48]

Since

$$\int \beta \operatorname{sech}(\beta t)\, dt = 2 \arctan\left(\tanh\left(\frac{\beta t}{2}\right)\right), \; \int_{-\infty}^{\infty} \beta \operatorname{sech}(\beta t)\, dt = \pi,$$

the middle axis $\boldsymbol{j}$ "swings a semicircle" about Galois critical axis $\boldsymbol{v} = (\cos t, \sin t, 0)$, which, in turn, is rotating uniformly about the fixed angular momentum $\boldsymbol{m}$. Note that w (that is, the magnitude of $\boldsymbol{w}$) attains its maximum at $t = 0$. There, we have

$$\boldsymbol{w}(0) = (\sigma\beta,\, 0,\, 1)\,, \; w(0) = \delta.$$

The matrix $Q(0, \beta, 1)$ is the identity matrix, so at $t = 0$ and $\sigma = 1$ the (ordered) moving basis $\{\boldsymbol{v},\, \boldsymbol{j},\, \boldsymbol{n}\}$ coincides with the fixed basis $\{\boldsymbol{u}_-,\, \boldsymbol{u}_+,\, \boldsymbol{u}\}$, as exhibited in figure 1. Permanent (unstable) rotation is not the only solution for vanishing b, contrary to (all) standard textbooks on mechanics. Most importantly, adding (to permanent rotation about the middle axis) the solution represented by the transition matrix Q, for a fixed σ, is not a full remedy, but only a half. The other half is represented by a matrix dual to Q, as σ flips its sign. The dual solutions, augmenting permanent rotation, are reminiscent of Abrarov's critical solutions, augmenting the unstable equilibrium of a simple pendulum [3].[49] The dual solutions are exhibited in figure 2.

6 A moral: Galois heritage ought not again be underestimated!

Discussing "the motion of a rigid body, in the absence of outside forces", in [6, p. 146],[50] Arnold writes that

> "The second revolution will be exactly like the first; if $\alpha = 2\pi p/q$, the motion is completely periodic; if the angle is not commensurable with 2π, the body will never return to its initial state."

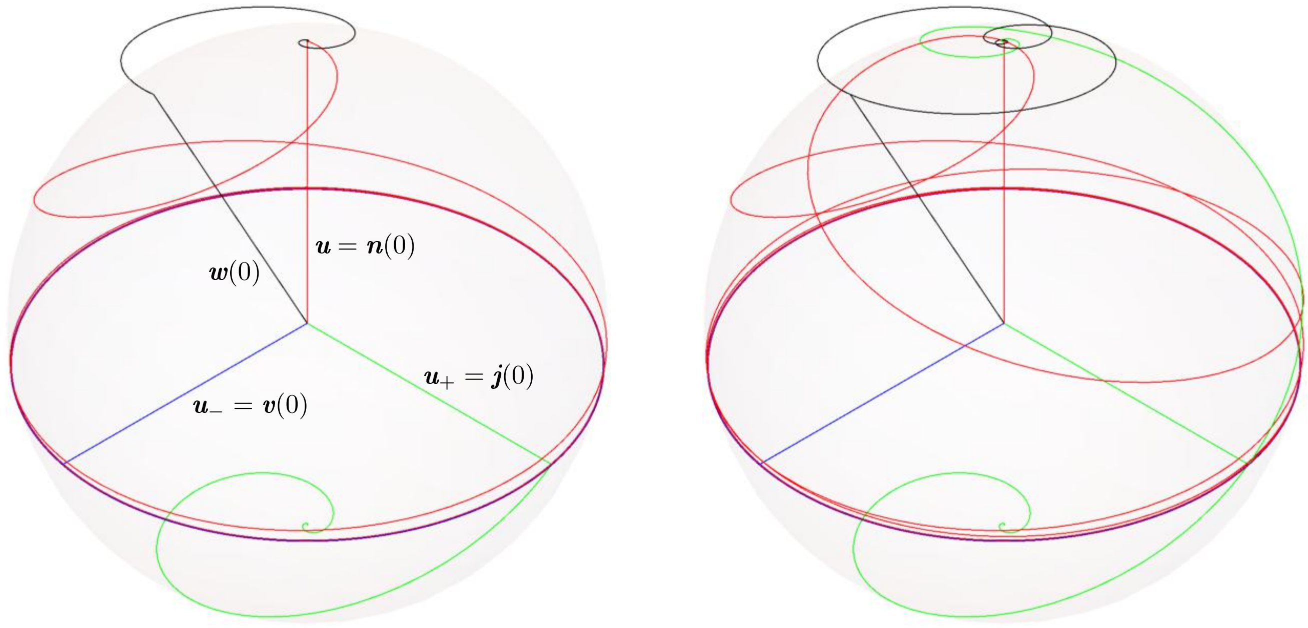
[46] We are no longer obliged here to match σ with the sign of $A - C$. Note that, in body frame, $\boldsymbol{w}_*(t) = -\boldsymbol{w}(-t)$.

[47] Exploiting the convention, just adopted, the (three) vectors of this inertial basis are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively.

[48] These coordinates might be calculated as coordinates of the vector, obtained by multiplying the matrix $\bar{Q}$ by the vector $w_1 \boldsymbol{v} + w_2 \boldsymbol{j} + w_3 \boldsymbol{n}$, that is, the vector $\boldsymbol{w}$ written with respect to the basis $\{\boldsymbol{v}, \boldsymbol{j}, \boldsymbol{n}\}$.

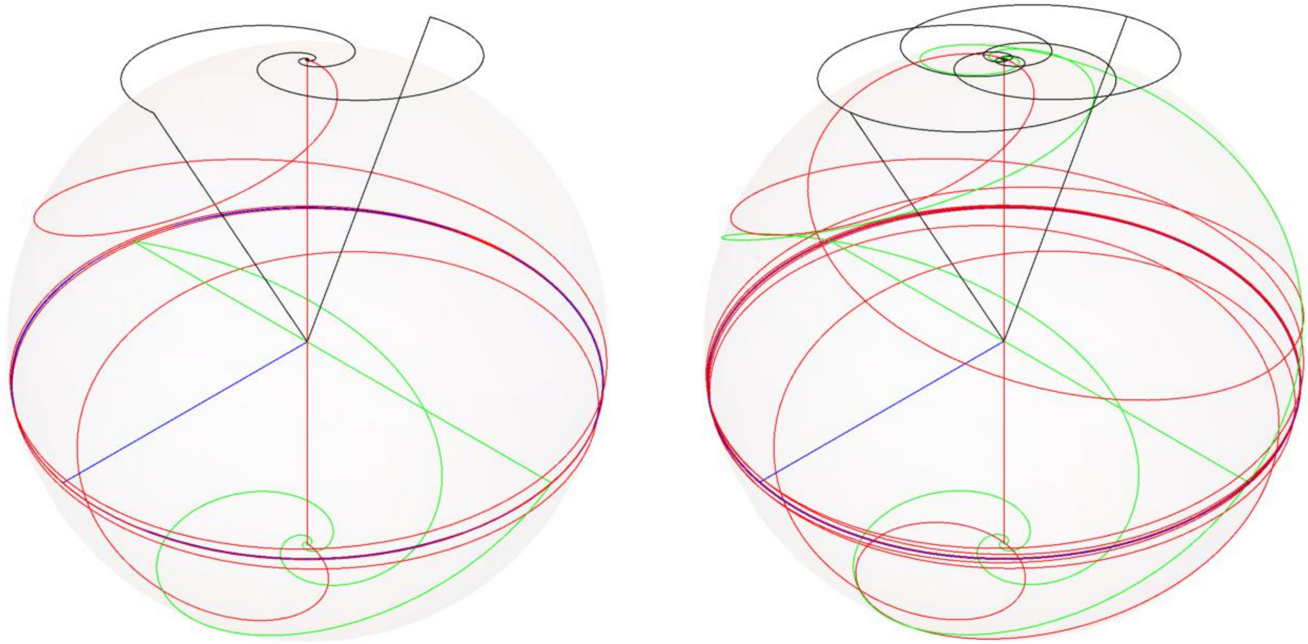
[49] The pendulum might "fall" from its upper (unstable) equilibrium position to one "side" or the other.

[50] The next (translated) statement appears on page 130 of the third Russian edition (1989).



$t \in [-8\,\pi, 0]$ $\qquad$ $t \in [-8\,\pi, 8\,\pi]$

Figure 1: The trajectories of the tips of the "rotating" vectors $\boldsymbol{v}$, $\boldsymbol{j}$, $\boldsymbol{n}$ in a König coordinate system, which basis vectors are $\boldsymbol{u}_-$, $\boldsymbol{u}_+$, $\boldsymbol{u}$. The rotating vector triple coincide with the "fixed" triple at $t = 0$. The corresponding colors are blue, green and red. Black is the color of the trajectory of the tip of the angular velocity $\boldsymbol{w}$. The angular velocity vector is (also) orthogonal to the (middle) vector $\boldsymbol{j}$ at $t = 0$. The chosen value for $\beta^2 = 5/16$ matches that for Dzhanibekov wingnut.



$t \in [-8\,\pi, 0]$ $\qquad$ $t \in [-8\,\pi, 8\,\pi]$

Figure 2: The trajectories for two dual solutions ($\sigma = \pm 1$), with $\beta^2 = 5/16$ (as in figure 1). Both solutions share one and the same Galois critical axis $\boldsymbol{v}$, which is shown in blue at $t = 0$. The axes $\boldsymbol{j}(0)$ and $\boldsymbol{n}(0)$ are reflected across the (invariant) axis $\boldsymbol{v}(0)$ as σ flips its sign.

Arnold is referring to Louis Poinsôt construction (of an ellipsoid rolling without slipping on an invariant plane), which (geometrically) describes the "trajectory" of the "tip" of the angular velocity of a freely moving rigid body (as the trajectory of the point of contact between the ellipsoid and the plane).[51] The said angle α is the angle of rotation of the body about the (fixed) angular momentum as the angular velocity vector returns to its initial state in body (rotating) frame. Of course, the need for calculating the angle α far exceeds the mere curiosity (concerning a single number-theoretic property of α) which Arnold (following others) chose to emphasize. Aside from implicitly presuming (unknown) integers, neither in the Russian nor in the English edition, did Arnold tell us what p and q were. He told us nothing more on calculating α.[52] Lev Landau, on the other hand, did not undermine the calculation of that angle, which (according to our notation) is the angle $4\,\psi(T_3)$, nor did he divert our attention to "philosophical" statements concerning rationality and eternity, but honestly (twice) admitted the "complexity" of calculating it in [17, p. 119],[53] and provided further reference [28]. One might go on to trace this issue back to the fundamental paper [16], dated March 17, 1850 by Jacobi, who (masterfully) calculated that angle as

$$\psi(T_3) = m \int_0^{T_3} \frac{\left(h - C r^2\right) dt}{m^2 - C^2 r^2},$$

and proceeded with clever manipulations, aiming at calculating the body rotation matrix. Exploiting the latter formula for $\psi(T_3)$, derived by Jacobi, we might deduce that

$$\psi(T_3) = \frac{T_3}{m}\left(\frac{m^2}{C} - c\,H_0\right), \; H_0 := \frac{2\,M(k_3)}{\pi}\int_1^{1/k_3} \frac{dx}{(b\,x^2/(b-c) - 1)\,\sqrt{(x^2-1)(1-k_3^2\,x^2)}} =$$

$$= N\left(k_3^2,\, 0,\, \frac{b}{2\,(b-c)},\, \frac{b}{b-c}\right) = N\left(\frac{1}{k_3^2},\, 0,\, \frac{a}{2\,(a-c)},\, \frac{a}{a-c}\right).$$

So, if $A = B$ then

$$H_0 = N\left(1,\, 0,\, \frac{b}{2\,(b-c)},\, \frac{b}{b-c}\right) = L\left(1,\, 0,\, \frac{b}{2\,(b-c)},\, \frac{b}{b-c}\right) = \frac{b-c}{c},$$

and (as we already know and expect) $\psi(T_3) = m\,T_3/B$.[54] Generally, however, Jacobi's integrand differs from ours,[55] and, in particular, it is not constant for vanishing b.[56] Jacobi did not construct Galois critical axis which alone is guaranteed to uniformly (and permanently) rotate about the (fixed) angular momentum if $b = 0$, even as the middle axis of inertia reverses its direction to either match or oppose the direction of the angular momentum,[57] and he did not address separating solutions which ought not be omitted before concluding calculations, especially, if calculations are to be (practically) implemented. Thereby, the "difficulties" (whether admitted, minimized or ignored) did, indeed, remain uneradicated from Jacobi's work. The (infinite) series (suggested by Jacobi) eventually (computationally) fail to converge as b approaches zero and critical cases of motion emerge.[58] Articles, such as [23], attempted to bridge the (seemingly mysterious) gap between seemingly "perfect" formulas and the on going (deeply rooted) practice of substituting them by "primitive" procedures for numerically solving (differential) equations of motion.[59] Other articles, such as [20, 8, 21], focused upon (a single yet significant issue of) providing alternative calculations or interpretations of the angle $4\,\psi(T_3)$.

The critical motion, discussed in the preceding section, corresponds to the separating solution which is missed by researcher who (innocently) presumed that the case with an unbounded (real) period (T_3) might safely

[51] Poinsôt ellipsoid is given via the (quadratic) equation, for the (three) components of the angular velocity, representing the preservation of energy. The invariant plane is orthogonal to the angular momentum.

[52] As was emphasized in [29], Poinsôt geometric interpretation enables neither a "complete" nor a time-dependent reconstruction of motion.

[53] The confession is (twice) made on page 155 of the fourth Russian edition (1988).

[54] Alternatively, we might observe that the equality $A = B$ implies the constancy of the integrand $(h - Cr^2)/(m^2 - C^2r^2)$ at $1/B$.

[55] Although, the values of the "complete" integrals "agree" with each other.

[56] Yet, as b approaches 0, H_0 approaches -1, as it should.

[57] Jacobi must had been aware of Galois letter [12], which Liouville published in 1846. Perhaps, Jacobi's early death (on February 18, 1851) precluded him from eventually acknowledging the outstanding significance of Galois contrubutions and their (unexpected) relevance for fully determining and most efficiently calculating the solutions to the problem that he (and Poinsôt) had addressed.

[58] The case here is not unlike approaching Abrarov critical motion from either oscillatory or rotary mode of motion of a (simple) pendulum [3].

[59] Such numerical methods "quickly" cumulate errors and are "blind" to critical solutions. Certainly, they would not detect solutions, aside from permanent rotations (without flipping), to our critical case with $b = 0$. On the other hand, they need not take into account the sign of b, as more sophisticated procedures require.

be ignored.[60] We are now being vividly reminded of this omission by Dzhanibekov's marvelous observation and by Burke's ingenious "half-twist experiment". Still and incessantly undermining the significance of the separating solution, the authors of [22, p. 401] had calculated the three projections of the angular momentum upon the axes of inertia merely up to a sign.[61] They did not indicate that among the eight possibilities, which emerge, only four satisfy the Euler equations. The four solutions correspond to four semicircles, arising from intersecting a triaxial ellipsoid with a concentric sphere which radius coincide with the middle axis of that ellipsoid.[62] These solutions might further be divided into two pairs. A pair corresponds to two coplanar semicircles, glued together to form a circle. Such a solution pair was explicitly presented as the dual solutions, corresponding to two signs of σ.[63] The issue of duality of solutions here closely resembles an analogous issue with the simple pendulum, as discussed in [3]. There is a loss of uniqueness at the position of unstable equilibrium of the pendulum, which must be augmented with Abrarov (two) critical separating solutions.[64] So is the case here, where (unstable) permanent rotation must be supplemented by two critical separating solutions, given by the (orthogonal) matrix Q (for two signs of σ). Without including these critical solutions, the (fundamental) problem of rigid body free motion is not entirely solved, so in accordance with the principle "Nil actum reputans si quid superesset agendum", emphasized by Gauss in [13, p. 629], it was not at all ever solved! May all and every credit for (finally) solving it be rightfully and entirely attributed to Évariste Galois![65]

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[60] As we now know, this "fundamental" omission is intrinsic(!) to the construction of Poinsôt (as the sliding of the ellipsoid was excluded).

[61] The authors seem oblivious to the significance of picking the "right" signs. Certainly, they must have never heard of Gauss four year struggle with the sign of his quadratic sum, described in a letter to Olbers (dated September 3, 1805), nor they ever seen Anna Johnson sign formula which we already mentioned in footnote 11. Of course (with such sloppiness), the authors never departed from body frame throughout their article. And, like many others, once they determined an infinite period of the separating solution they lost every interest in it!

[62] An enlightening letter, given in [14], from Sir William Rowan Hamilton to the Reverend Charles Graves is recommended here.

[63] A first presentation of these dual solutions was delivered by the author of this paper on October 26$^{\text{th}}$, 2016 at "the Egorov seminar on the mechanics of space flight" (conducted at the Moscow State University by Victor Sazonov).

[64] Contrary to common belief (refuted by Dmitry Abrarov), the pendulum at the unstable equilibrium does not require any push (however small) in order to yield a separating solution. In other words, no unique single-valued function represents a solution to the unstable equilibrium of a simple pendulum. In particular, the (full) solution cannot be limited to a constant function, representing a "standing" pendulum. Two additional solutions correspond to (full) rotations (in infinite time) in either (clockwise or counterclockwise) direction [3].

[65] Informally, yet eloquently, put by Nikolay Vavilov (PDMI, St. Petersburg): "There are tens of thousands of mathematicians like Cauchy, but Galois is one of his kind!"

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