Periodic motions of the mechanical system with one degree of freedom

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The problems of stability and stabilization of motion. M.: CC RAS. 1999. P. 100-112.

The English translation

The unified approach to investigation of oscillatory and rotational motions of a mechanical system with a single degree of freedom is proposed. The problem of planar periodical oscillatory and rotational motions of a satellite is proposed as an application. The satellite orbit is supposed to be an elliptical. The satellite is exposed to the action of the torque of a gravitational and aerodynamical origin. An additional disturbing factors are taken into account also.

The problem of mathematical pendulum includes the motions of two sorts: oscillations around the lower equilibrium position and also rotations in one or another direction. The pendulum angle deviation from the equilibrium position and the angle velocity are periodic functions of time for oscillatory movements. In this sense the oscillation is the periodic solution of the corresponding dynamical equation. Oscillations are represented by closed trajectories on the phase plane while rotational motion trajectories are not closed. The angle of rotational motion is the sum of the periodical function and linear function of time. Hence the rotational motion is not a periodical solution of the pendulum equation. Such solutions are named [1] generalized periodical solutions.

Independently of how solutions describing periodical rotational movements are called, the problem of development of the theory of such solutions arises. It turns out that both oscillatory and rotational motions are described by common unified theory, if closure conditions of the integral curve in the proper space are considered as conditions of periodicity. This space is the phase cylinder in the frame of mathematical pendulum problem because both oscillatory and rotational motions have the closed trajectory on the phase cylinder. On this cylinder the system state is defined as the angular coordinate (the angle of pendulum rotation) and angle velocity measured along the cylinder axis. The oscillatory motion on the phase cylinder is considered as a special case of rotational motion, when the number of representative point rotations around the cylinder along the closed curve is zero.

The system with single degree of freedom only is investigated in this paper. The problem of periodical oscillatory motions of the satellite in the elliptic orbit plane is considered. The gravitational and aero-dynamical torques are supposed to act on the satellite as well as disturbing factors.

Note that general results on the periodic rotational motions theory will be given in the separate paper. Investigation of the satellite oscillations due to the action of the gravitational and aero-dynamical torques only is in [2].

1. Periodical motions. Let us consider the system with the single degree of freedom

$$\ddot{z} = Z(z, \dot{z}, t), \tag{1}$$

where function $Z = Z(z, \dot{z}, t)$ is smooth, 2π-periodical with respect to z and t.

Definition. The solution $z=\varphi(t)$ equation (1) is called a $2\pi k$-periodical motion ($k \in N$) if

$$\varphi(t + 2\pi k) = \varphi(t) + 2\pi m, \qquad (m \in Z). \tag{2}$$

The speed $\dot{\varphi}(t)$ will be $2\pi k$-periodical function of t, and function $\varphi(t)$ takes the form

$$\varphi(t) = (m/k)t + \psi(t), \quad \psi(t + 2\pi t) = \psi(t).$$

On the phase cylinder $(z, \dot{z})$ the integral curve will be closed in a time $\Delta t=2\pi k$. If $m=0$ then motion is oscillatory, if $m \neq 0$ then motion is rotational. If $m>0$ we have direct rotation, if $m<0$ we have inverse rotation.

On the other hand function $\varphi(t)$ satisfies the condition (2) if its integral curve will be closed on the phase cylinder in a time $\Delta t=2\pi k$. Hence the necessary and sufficient conditions of the $2\pi k$-periodical motion existence have the form

$$\varphi(t_0 + 2\pi k) = \varphi(t_0) + 2\pi m \qquad (m \in Z), \qquad \dot{\varphi}(t_0 + 2\pi л) = \dot{\varphi}(t_0) \tag{3}$$

(t_0 - initial time moment).

2. The problem of the continuation of rotational motion with respect to a parameter. Let us consider the equation sufficiently smooth and $2\pi k$-periodical with respect to z and t

$$\ddot{z} = Z(z, \dot{z}, t) + \mu Z_1(\mu, z, \dot{z}, t), \tag{4}$$

where μ is a small parameter. Suppose that the generating equation (1) admits the $2\pi k$-periodical motion (2). Let $\mu \neq 0$ and consider the existence problem of such a $2\pi k$-periodical motion of equation (4) which converts into the motion (2) when $\mu \to 0$.

The necessary and sufficient conditions of the $2\pi k$-periodical motion existence have the form

$$z(\mu, z_0, \dot{z}_0, t_0, t_0 + 2\pi k) = z_0 + 2\pi m \qquad (m \in Z), \qquad \dot{z}(\mu, z_0, \dot{z}_0, t_0, t_0 + 2\pi k) = \dot{z}_0 \tag{5}$$

($(z_0, \dot{z}_0)$ - initial point at the time moment $t=t_0$).

The system of functional equations (5) has the solution $z_0 = \varphi(t_0), \dot{z}_0 = \dot{\varphi}(t_0)$ if μ=0. Therefore, the implicit function theorem guaranties that system (5) is consistent for sufficiently small $|\mu| \neq 0$ if the following condition is satisfied

$$\det\begin{Vmatrix} \partial z(0,z_0,\dot{z}_0,t_0,t_0+2\pi k)/\partial z_0 - 1 & \partial z(0,z_0,\dot{z}_0,t_0,t_0+2\pi k)/\partial \dot{z}_0 \\ \partial \dot{z}(0,z_0,\dot{z}_0,t_0,t_0+2\pi k)/\partial z_0 & \partial \dot{z}(0,z_0,\dot{z}_0,t_0,t_0+2\pi k)/\partial \dot{z}_0 - 1 \end{Vmatrix} \neq 0 \tag{6}$$

(the calculations in (6) are carried out for $z_0 = \varphi(t_0), \dot{z}_0 = \dot{\varphi}(t_0)$).

Let us consider the system in variations for the generating equation (1) for the solution (2). The condition (6) means that the characteristic equation of this system in variation has no the unit root or in other words, this system has no periodical solutions. Hence we have the Poincare-isolated root by analogy with the periodical (in usual sense of the word) motion (m=0).

Theorem 1. In the Poincare-isolated case (6) the equation (4) for sufficiently small $|\mu| \neq 0$, has a unique 2πk-periodical motion which tends into the motion (2) as $\mu \to 0$.

3. Reversible system. Let us suppose that the considered equation (1) is reversible, i.e. that function Z satisfies the condition $Z(z,\dot{z},t) = -Z(-z,\dot{z},-t)$. In this case consider the periodic motion

$$z(\pi\beta,\dot{z}_0,\pi\alpha,-t+\pi\alpha) = -z(\pi\beta,\dot{z}_0,\pi\alpha,t+\pi\alpha) + 2\pi\beta, \quad \dot{z}(\pi\beta,\dot{z}_0,\pi\alpha,-t+\pi\alpha) = \dot{z}(\pi\beta,\dot{z}_0,\pi\alpha,t+\pi\alpha),$$

$$z(\pi\beta,\dot{z}_0,\pi\alpha,\pi(\alpha+2k)) = -z(\pi\beta,\dot{z}_0,\pi\alpha,\pi\alpha) + 2\pi m, \qquad (\alpha,\beta,m \in Z) \tag{7}$$

symmetric with respect to fixed set $M = \{t,z,\dot{z}:\ \sin(t)=0, \sin(z)=0\}$.

It turns out that necessary and sufficient conditions of existence of symmetric 2πk-periodic motion has the form

$$z(\pi\beta,\dot{z}_0,\pi\alpha,\pi(\alpha+k)) = \pi(m+\beta). \tag{8}$$

The condition (8) means that the fixed set M is crossed at the time moments $\pi\alpha$, $\pi(k+\alpha)$. The relation (8) gives the method for construction of all symmetric, 2πk-periodical motions (both oscillatory and rotational) of the reversible system.

Consider the four components of the set M:

$$M_{00} = \{t,z,\dot{z}:\ t=0, z=0\}, \qquad M_{01} = \{t,z,\dot{z}:\ t=0, z=\pi\},$$

$$M_{10} = \{t,z,\dot{z}:\ t=\pi, z=0\}, \qquad M_{11} = \{t,z,\dot{z}:\ t=\pi, z=\pi\}.$$

Let the $M^{\pi}_{ij}(i,j=0,1)$ be the image of the set M_{ij} when t is varying from $\pi\alpha$ to $\pi(k+\alpha)$, and α=0 or α=1. Then intersection of sets $M^{\pi}_{oo}, M^{\pi}_{o1}, M^{\pi}_{1o}, M^{\pi}_{11}$ with the fixed set M defines all points which belong to the symmetric, 2πk-periodical motions for $t=\pi(k+\alpha)$.

This method allows to construct all periodic motions, is convenient for numerical implementation, and has been used for investigation of the problem of the satellite motions along elliptic orbit plane [2-4].

Suppose now that disturbance Z_1 in equation (4) relates to the reversible functions too, i.e. the equality is true $Z_1(\mu,-z,\dot{z},-t) = -Z_1(\mu, z, \dot{z}, t)$. Then the problem of the continuation of symmetric 2kπ-periodical motion with respect to a parameter μ has the solution. This fact is followed from the relation (8) if the condition is true

$$\partial z\big(\pi\beta, \dot{z}_0, \pi\alpha, \pi(\alpha + k)\big) / \partial \dot{z}_0 \neq 0 . \tag{9}$$

Notice that the condition (9) is more weaker then the condition (6). In particular this condition is always true for the rotational motions of the reversible system, which are close to the conservative system with single degree of freedom.

Let us consider the question concerning to the stability of symmetric periodical motion. Owing to the symmetry of the solution and also to 2kπ-periodicity equation (1) with respect to z and t, the equation of the perturbed motion will be reversible too and 2kπ-periodical with respect to t. Therefore the stability of linear approximation guaranties the Lyapunov stability almost always (one coefficient of the normal form is not equal to zero) if there are no resonance up to the forth order including [5].

Suppose now that the symmetric 2kπ-periodical motion of the generated equation (1) is unstable with respect to linear approximation, or has stable linear approximation and there is no resonance till the forth order inclusive. Then the condition (9) is true and there exists the symmetric 2kπ-periodical solution of the system (4) if parameter μ is sufficiently small $|\mu| \neq 0$. This solution takes the stability property of the linear approximation and there is no mentioned resonance again. Accordingly to Yu. N. Bibikov [5] this fact follows the Lyapunov stability.

Theorem 2. Let the reversible generated system (1) has the symmetric 2kπ-periodical motions which are unstable with respect to first-order approximation or are stable with respect to linear approximation. Then these motions have the unique continuation with respect to a parameter μ in the class of reversible perturbations if parameter μ is sufficiently small $|\mu| \neq 0$. Besides (in the sense of initial values measure for the periodical motion) stable motions (with respect to linear approach) give the Lyapunov stable motions.

4. Characteristic indexes of reversible system. The characteristic indexes problem rises as it follows from above when the stability of the periodic motions is investigated. Moreover this problem must be solve if the condition (6) is checked up when task of the continuation of symmetric periodical motion of reversible generating system (1) in the class perturbations of the general view $\mu Z_1(\mu, z, \dot{z}, t)$.

Every symmetric periodical motion of reversible system is built numerically on the base of method above (section 3). Therefore the characteristic indexes calculation process must be parallel to the Cauchy problem solving for the equation (1) and for the equation

$$\delta\ddot{z} = (\partial Z / \partial z)_* \delta z + (\partial Z / \partial \dot{z})_* \delta\dot{z}. \tag{10}$$

The sign * means that the functions defined the symmetrical periodical motion substitute the values $z, \dot{z}$ in the partial derivations.

The linear reversible systems are analysed in details [2,7]. It turns out that the unique solution is enough for defining of the characteristic indexes of equation (1).

Thus the algorithm of characteristic indexes calculating is reduced to the following. The Cauchy problem is solved for the system (1),(10) on interval $[0,2\pi k]$ with initial value $z_0 = \pi\alpha,\ \dot{z}_0 = \dot{z}_*,\ \delta z_0 = 1,\ \dot{z}_0 = 0$ ($z_0 = \pi\alpha,\ \dot{z}_0 = \dot{z}_*$ assign the initial point for the symmetric 2πk-periodic motion). Then the characteristic indexes λ is calculated according to formula

$$\lambda = \pm \frac{1}{2\pi k} \operatorname{arcch}\big(\delta z(2\pi k)\big).$$

Theorem 3. Let the reversible generating system (1) has the symmertic 2πk-periodic motion and $\lambda \neq ip$, $(p \in N)$. Then these motions is continued with respect to parameter p in the class of general view.

5. Satellite on the elliptic orbit under action of gravitational and aerodynamical torques. The satellite motion in the elliptic orbit plane under action of gravitational and aerodynamical torques is described by the equation [8]

$$\begin{aligned} &\frac{d^2\alpha}{d\nu^2} + \chi(1+e)^2 \frac{\rho^* \sqrt{1 - 2e\cos\nu + e^2}}{\rho^*_\pi (1 + e\cos\nu)^4} [\sin\alpha + e\sin(\alpha+\nu)] - \\ &- 2\left(1 + \frac{d\alpha}{d\nu}\right)\frac{e\sin\nu}{1 + e\cos\nu} + \mu\frac{\sin\alpha\cos\alpha}{1 + e\cos\nu} = 0 \end{aligned} \tag{11}$$

Here α is the angle between satellite centre mass radius-vector and one of the inertia axis, ν is the true anomaly of satellite orbit position, e is eccentricity of elliptical orbit of the satellite centre mass, μ is satellite inertial parameter ($|\mu| \le 3$), χ is the constant aerodynamical parameter characterised the atmosphere action ($\chi \ge 0$), ρ^*_π is the atmosphere density in the orbit perigee point.

The atmosphere density ρ^* depending on altitude is given the tabular form. The satellite centre mass radius-vector is even function of the true anomaly. Therefore the relation $\rho^*(\nu)$ is even function too and is 2π-periodical function of ν.

One can see that the equation (11) is invariant with respect to the replacement $(\nu, \alpha, \dot{\alpha}) \rightarrow (-\nu, -\alpha, \dot{\alpha})$, i.e. the equation is reversible one with fixed set $M = \{\nu, \alpha, \dot{\alpha} : \sin(\nu) = 0, \sin(\alpha) = 0\}$.

The equation (11) is obtained under a few simplifying assumptions (the atmosphere is nonmoving with respect to the Earth, the pressure centre lies on the main satellite inertia axis) [10]. Besides the relative satellite motion is affected by another torque: sunshine pressure torque, magnetic torque, aerodynamical dissipative torque etc. The relation $\rho^*(\nu)$ is given with certain accuracy too.

The facts above are followed that it is necessary to take into account the additional small disturbing factors in the right side of equation (11) during the dynamical investigation of the satellite under gravitational and aerodynamical torques. Some of these factors relate to the class of reversible one (deviation from the accurate relation $\rho^*(\nu)$, sunshine pressure torque), other factors have the general nature.

The periodical motions of equation (11) are studied in [2,8,11]. The equation (11) takes the form of equation for the conservative system with one degree of freedom when the orbite is circular (e=0, $\rho^*=\rho^*_\pi$). This equation was analysed [11], the all generating 2π-periodical oscillatory motions are found [8]. All periodical rotational motions are generated when orbit is circular (section 3). These motions are found. At the same time the reversible 2π-periodical rotational motions are corresponded to the oscillatory motions in the immovable coordinate system. These motions are interesting from the viewpoint of orientation problems.

The results [11] were used [8] for the calculating of the initial angular velocities $\dot{\alpha}(0)$ along 2π-periodical oscillatory motions and also for stability analysis of these motions (with respect to first-order approximation). The method (section 3) was utilized [2] for the finding of all 2π-periodical (forward and backwards) rotations under diverse magnitudes of the aerodynamical parameter χ and also for the stability analysis of these motions (section 4).

The theorem 2,3 application of the satellite dynamics admits the following conclusion.

Theorem 4. Almost all 2π-periodical oscillatory motions and rotational flat motions of the satellite under gravitational and aerodynamical torques generate the same motions under additional 2π-periodical (with respect to α and ν) perturbation factors. Almost all (in the sense of the measure of the parameter (e, μ, χ) space) stable (with respect to first-order approximation) periodical motions generate the Lyapunov stable motions.

This research was supported by the Russian Foundation for Basic Research (96-15-96051, 97-01-00538) and the "University of Russia" Programme.

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