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MATHEMATICS

Some Properties of One-Dimensional Unimodal Mappings

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One-dimensional unimodal mappings (ODUMs) are a popular object illustrating the plenitude of dynamical modes in simple difference equations. The related results include Sharkovskii's order and doubling cascades [1, 2]. This paper studies ODUMs which arise in describing the dynamics of animal populations. For ODUMs of this type, there exists a scenario for varying a distinguished parameter, for which stability zones with stable cycles successively emerge. Inside a stability zone, cycles have constant period; periods in successive zones form a series of consecutive positive integers 1, 2, 3, 4, ... Stability zones are separated from each other by transition zones with more complicated regimes. We suggest original methods for finding periodic trajectories of ODUMs.

THE ORIGIN OF THE PROBLEM

In our research, the problem of studying properties of ODUMs (for difference equations) arose in describing the dynamics of animal populations in the framework of mathematical models of tundra habitations and communities [3–5]. The study performed has resulted in constructing a set of interrelated models based on detailed simulation models constructed in collaboration with biologists on the basis of expert estimates for dependences describing seasonal fluctuations of parameters. An analysis of the results of computational experiments on complementary models on the plants–lemmings–arctic foxes community (see Fig. 1a) and a lemming population with the age structure taken into account has led to a simplified model in the form of a difference equation whose graphical representation is given in Fig. 1b. Using the difference equation

$$X^{t+1} = F(X^t), \qquad (1)$$

which relates the normalized population X^t of lemmings in two consecutive years, we were able to reproduce a time dynamics qualitatively close to that of real populations of lemmings [3–5]. The existence of such difference equations has qualitatively changed the technology and possibilities of modeling. A parametric study of these equations has made it possible to determine domains of parameters in the initial simulation models that ensure dynamical regimes close to those observed in the nature. Using difference equations, we were to formulate conjectures concerning the mechanisms of variations of tundra animal populations and distinguish three principal parameters determining them, namely, (i) the rate of increment of biomass in a favorable year; (ii) maximum population; (iii) survivability under most unfavorable conditions (or two dimensionless parameters, the relative rate of population increment and the fraction of surely surviving animals). The first parameter characterizes the balance between natality and mortality in the absence of "environmental pressure," the second characterizes the ecosystem as a whole and reflects the coevolution of lemmings and forage reserve, and the third characterizes the adaptation abilities of lemmings under extremal conditions and is largely determined by local characteristics, such as relief in regions of wintering.

This of ODUMs is of special interest due to the fact that, for the population of lemmings in West Taimyr being modeled, the occurrence of maximum population every three years is typical. At the same time, the presence of a cycle with period 3 in Sharkovskii's order ensures the existence of cycles any length [1]. In this paper, we suggest an ODUM type and a variation scenario for a distinguished parameter under which cycle periods change so as to form the series of positive integers.

Statement. *In the ODUM*

$$X^{t+1} = F_1(X^t) \equiv \begin{cases} 2X^t, & 0 \le X^t \le 0.5 \\ d\ (0 < d < 1), & 0.5 < X^t \le 1 \end{cases} \qquad (2)$$

with decreasing the parameter d from 1 *to* 0, *globally stable cycles of period n, where n runs over all positive integers, successively arise.*

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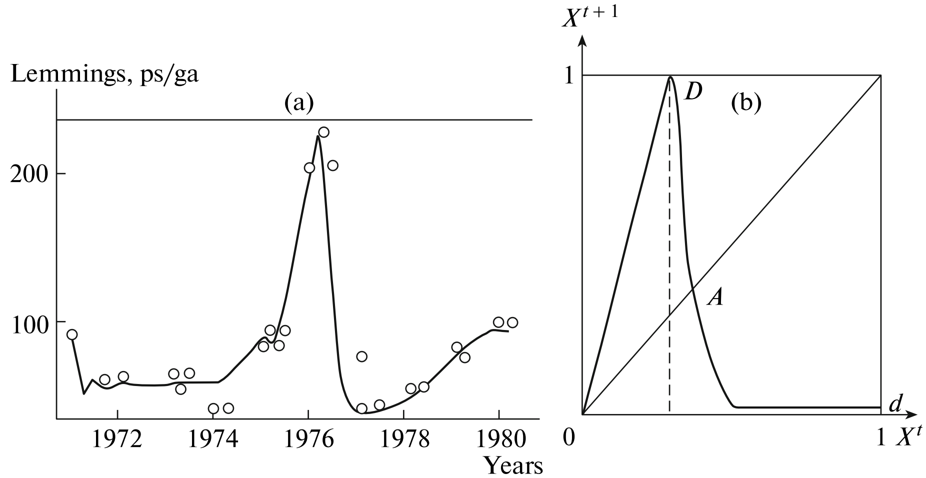

Fig. 1. (a) The dynamics of the lemming population computed from results of a simulation experiment with the plants–lemmings–arctic foxes model (solid line) and the dynamics of the hoofed lemming population recorded on the Vrangelya island (circles) [6]; (b) the graphic representation of the difference equation $X^{t+1} = F(X^t)$, which relates the normalized lemming populations X^t in two successive years, based on an analysis of models for tundra populations and communities. Here, D is the point of maximum population, A is the equilibrium population, d is the population of lemmings in an optimal biotope (the notion of an optimal biotope was introduced in [3, 5] as a habitat with optimal conditions; in an optimal biotope, a certain number of animals survives under any conditions).

The proof of this statement is given in [4].

The above statement deals with the discontinuous function $X^{t+1} = F_1(X^t)$. Let us remove discontinuity. In a neighborhood of the point of discontinuity, we join the left and right parts of the graph of F_1 by a monotonically decreasing curve close to a vertical segment. For example, consider the function F_1 defined by (for $r \gg 1$)

$$X^{t+1} = F_1(X^t) \equiv \begin{cases} 2X^t, & 0 \le X^t \le 0.5 \\ 1 - r(X^t - 0.5), & 0.5 < X^t \le 0.5 + \dfrac{1-d}{r} \\ d(0 < d < 1), & 0.5 + \dfrac{1-d}{r} < X^t \le 1. \end{cases} \quad (3)$$

For such a function, computational experiments were performed. The results of some of them are shown in Fig. 2. As above, the character of dynamical regimes was studied under the variation of the parameter d from 1 to 0. In this case, the dynamical regimes become more diverse. In Fig. 2, stability zones separated by transition zones with complicated regimes (black vertical strips) are distinguished. Inside the stability zones, the period of trajectories is constant, and in passing from zone to zone, the period takes consecutive positive integer values. In each of the transition zones, there exist periodic trajectories with period exceeding any given positive integer. The width of the transition gaps can be rendered arbitrarily small by letting the parameter r tend to infinity.

There is a certain correspondence between the presence of transition zones and the recorded dynamics of real populations. In the absence of a distinct three-year cycle (in regions warmer than Taimyr), two- and five-year intervals between population peaks occur [6].

METHODS FOR STUDYING PERIODIC TRAJECTORIES

In this paper, we suggest constructive methods for detecting periodic trajectories of ODUMs. Consider the ODUM

$$X^{t+1} = F(X^t),$$

which maps the interval [0, 1] to itself, with an equilibrium at a point A.

Suppose that the function F monotonically increases on an interval $[0, D]$ with $D < 1$ so that $F(X^t) > X^t$, attains its maximum value $F(D) = 1$ at the point D, and then monotonically decreases (possibly, it eventually goes horizontally; see Fig. 1b).

To analyze the behavior of trajectories, we define the two sets of points $M = \{A_n, n = 0, 1, 2, ...\}$ and $K = \{D_n, n = 0, 1, 2, ...\}$. The set M (K) consists of points $A_i(D_i)$ for which $F^i(A_i) = A$ (respectively, $F^i(D_i) = D$) (see Fig. 3), where $F^i(\cdot) = F(F(...(F)))$ is the ith iteration of the mapping under consideration. We set $A_0 = A$ ($D_0 = D$). If the trajectory is on the left of the point A and the ordinate of one of its points falls in the interval $[A_i, A_{i-1}]$, then, in the next pass, the trajectory falls

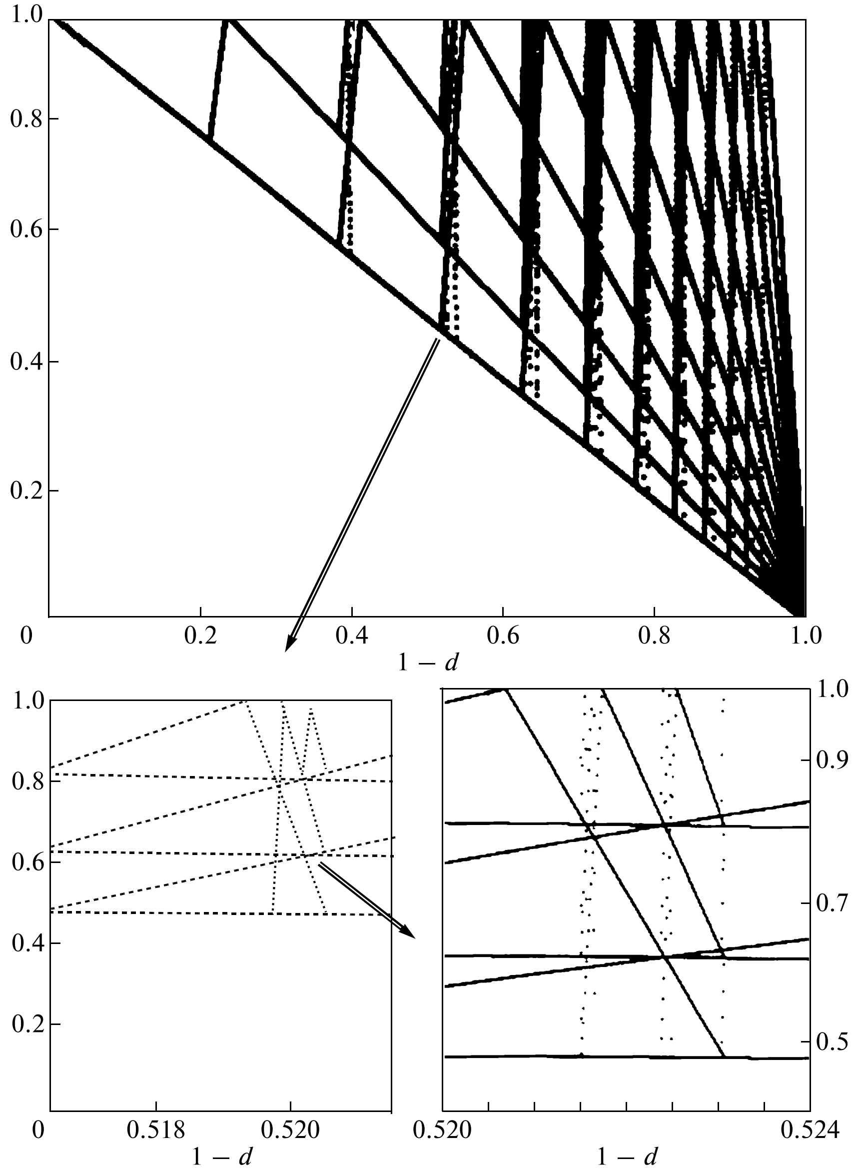

Fig. 2. Results of computational experiments on model (3): the dependence of trajectories on $1 - d$. On the abscissa axis, the value $1 - d$ is marked. The vertical section of the graph at a chosen value of d consists of trajectory points.

in the interval $[A_{i-1}, A_{i-2}]$. Thus, the above statement can be reformulated as follows.

Statement 1. *If the function* F_1 *defined by* (2) *satisfies the condition* $A_n < d < A_{n-1}$, *then there exists a unique globally stable cycle of period n* for $n = 1, 2, 3, 4, \dots$.

In studying ODUMs, a standard trick is the consideration of the ith iterations $F^i(\cdot) = F(F(\dots(F)))$ of the mapping F. In this paper, to study properties of periodic solutions, we introduce two constructions, a return line (RL) and a mapping beyond the equilibrium (MBE). These constructions are related to the fact that, for the ODUM under consideration, the equilibrium point A breaks the interval [0, 1] into two parts, $[0, A]$ and $[A, 1]$. These parts are nonequivalent. The trajectory cannot be contained in the second part during two successive passes.

Definition 1. The MBE is defined as the self-mapping of the interval $[A, 1]$ taking each point X from this interval to the first return point Y of the trajectory beyond the equilibrium.

The new mapping can be investigated by usual methods (finding stationary points, considering nth iterations, etc.). The interval $[A, 1]$ can be linearly transformed into the interval [0, 1]. In addition to the MBE, we define the return line RLn of order n.

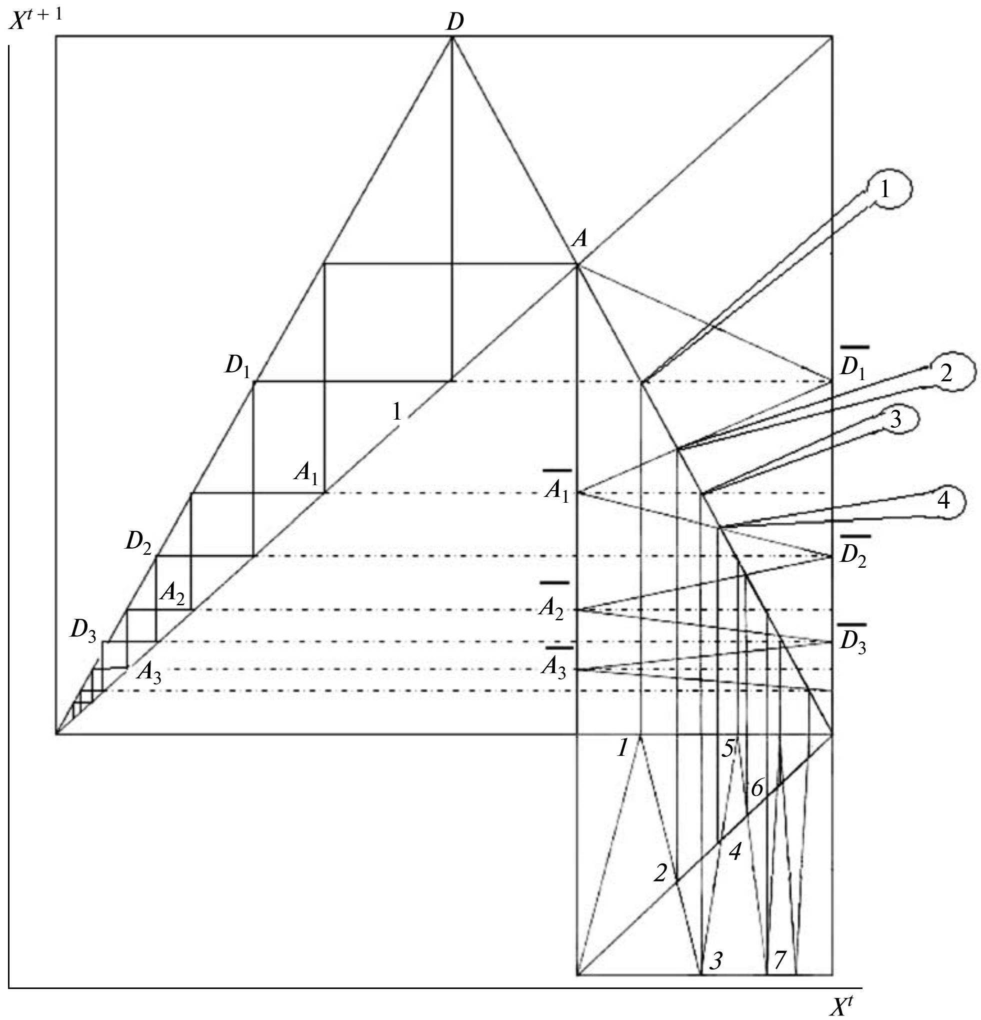

Fig. 3. A graphic representation of the triangular mapping, its RL1 $\bar{A}$ $\overline{D_1}$ $\overline{A_1}$ $\overline{D_2}$ $\overline{A_2}$ $\overline{D_3}$ $\overline{A_3}$ and MBE (the latter is shown in the square displayed at the lower right corner). The points *1* and *5* are generated by the preimages of the point *D*, the points *3* and *7* are generated by the preimages of the equilibrium point *A*, and the points *2, 4*, and *6* are generated by the intersection of the graph of the triangular mapping with the RL1 and determine periodic trajectories (one trajectory with period 2 and two trajectories with period 3).

Definition 2. The *n*th-order return line RL*n* for the mapping F is the curve in the rectangle $A \le X^t \le 1$, $0 \le X^{t+1} \le A$ being the graph of the function $F_c^{(n)}(X^{t+1})$ which maps the interval $0 < X^{t+1} \le A$ to the interval $A \le X^t \le 1$ by the algorithm described below.

Algorithm for constructing RL*n*. Through any point X^{t+1} from the interval $0 \le X^{t+1} \le A$ in the rectangle $A \le X^t \le 1$, $0 \le X^{t+1} \le A$ we draw a horizontal line. Its intersection with the graph of the function $F(\cdot)$ determines initial conditions for constructing the corresponding trajectory. We construct this trajectory by the Lamerey staircase algorithm. According to this algorithm, for the *n*th return beyond the equilibrium, we draw the corresponding vertical line from the bisector of the angle between the abscissa and the ordinate axes. The intersection point of this line with the horizontal test line belongs to the RL*n* with coordinates (X^t, X^{t+1}).

Thus, in the rectangle specified above, each value X^{t+1} is assigned the value X^t; thereby, a function $X^t = F_c^{(n)}(X^{t+1})$ is defined. The graphs of such functions are exemplified in Figs. 3 and 4. The intersection points of RL*n* with the graph of the initial function F determine periodic trajectories. Using FP*n*, we can find all periodic trajectories with period at most *n*.

Statement 2. *A periodic trajectory is stable if it contains the intersection point of the graph of the mapping* $F(X^t)$ *with that of the function* $F_c^{(n)}(X^{t+1})$, *it has the cor-*

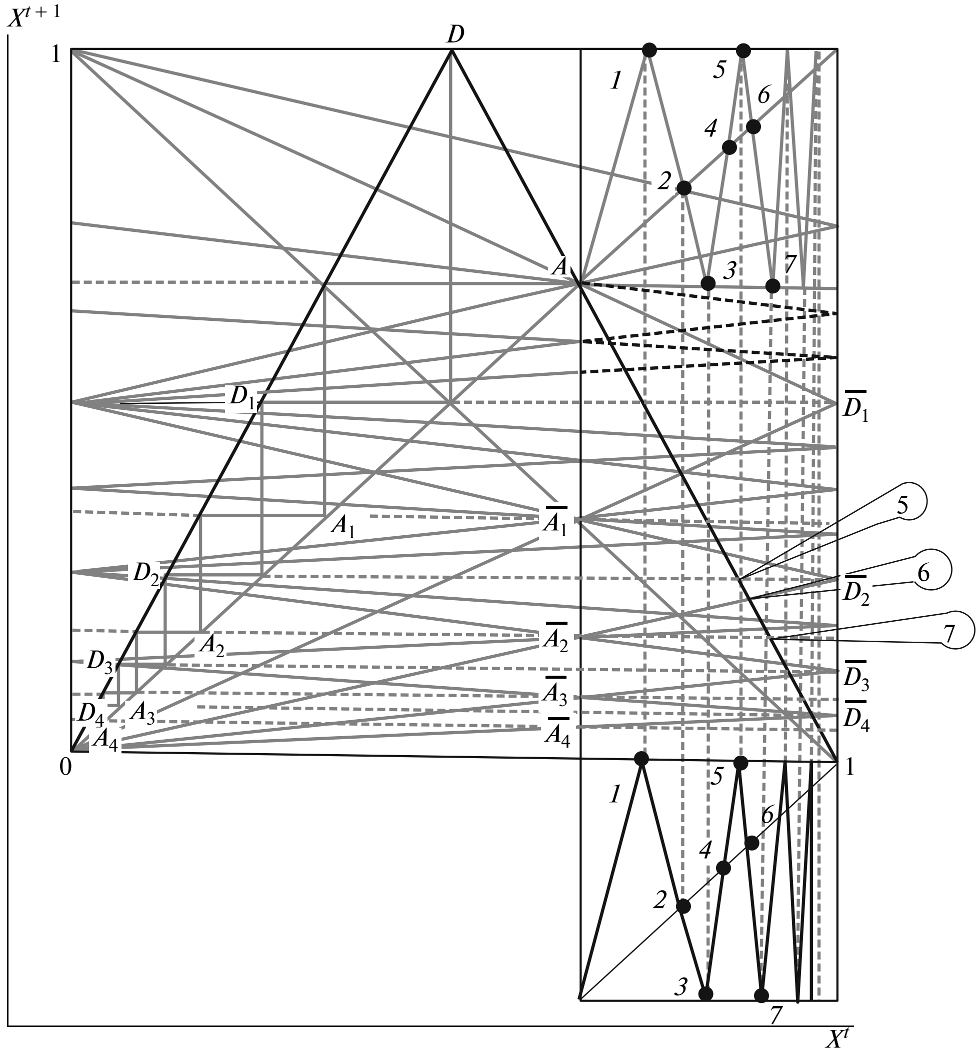

Fig. 4. A graphic representation of the triangular mapping, its RLn and MBE (the MBE is shown in the squares displayed at the lower right corner angle and at the upper right corner) and its nth iterations rotated through 90°. The RLn is formed by the corresponding fragments of these mappings, and the MBE at the upper right corner is constructed from fragments of the nth iteration of the initial mapping. The points labeled by digits coincide those shown in Fig. 3.

responding derivatives at this point, and these derivatives satisfy the condition $|F| < |(F_c^{(n)})'|$.

This statement is proved by the method of contraction mappings [4].

In the formation of the MBE and FP1 lines, a special role is played by the trajectories generated by the points $\{A_n, n = 0, 1, 2, \ldots\}$ and $\{D_n, n = 0, 1, 2, \ldots\}$. The former form the minima and the latter, the maxima of these lines. Both types of lines are saw-tooth curves; the number of teeth for the MBE and RL1 is determined by the number of preimages of the equilibrium. For triangular and logistic mappings, they form a countable set.

The construction of the RL1 and the MBE is exemplified in Fig. 3 for the standard triangular mapping

$$X^{t+1} = F_0(X^t) = 1 - 2|0.5 - X^t|.$$

The RL and MBE constructions described above are related to the standard ith iterations $F^i(\cdot) = F(F(\ldots(F)))$. The relationship between them is illustrated in Fig. 4, which shows that the MBE is formed by the fragments of $F^i(\cdot)$ occupying the upper right corner ($X^t, X^{t+1} > A$), and the RL is formed of the $F^i(\cdot)$ rotated through 90°. The conjecture on the possibility of constructing the RL and the MBE in this way has proved true for all mappings known to these authors (in particular, for the logistic mapping).

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