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Simulation of Suspended Substance Dispersion on the Ocean Shelf: Effective Hydraulic Coarseness of Polydisperse Suspension

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Abstract—Suspended substance dispersion in a water body is simulated in the case when the spread area is considerably larger than the depth of the water body. It is shown that, even if vertical turbulent exchange has a large effect, the problem of computing the evolution of polydisperse suspended pollutants produced by an instantaneous point source can be reduced to the integration of a few one-dimensional evolution problems and to the solution of one two-dimensional problem. This result can be used to design efficient solution methods for the practically important problem of computing the dispersion of a suspension produced by a time-continuous and/or spatially distributed source of water pollution.

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1. INTRODUCTION

Recently, interest in the computation of suspended substance dispersion in water bodies has increased as motivated by the necessity of performing assessments of various anthropogenic impacts on the environment. Such assessments are required, for example, in the construction of underwater pipelines and drilling platforms on the oceanic shelf, in dredging operations, etc. In principle, available regulations impose rather severe requirements on the quality of mathematical models used for such assessments. For example, according to these regulations, the total concentration of mineral suspension at control points located about 250–500 m away from a pollution source must not exceed 1 mg/l, while near the source this characteristic usually amounts to 10 g/l and more.

In the description of suspended substance transport, we can distinguish two qualitatively different areas, namely, a near-field region, whose length scale correlates with the size of a pollutant source (e.g., a water outlet structure, a dredger, etc.), and a far-field region (with control points deployed), whose size considerably exceeds the characteristic length of the near-field region.

In the near-field region, the concentrations of suspended substances are high and the simulation of their transport generally has to be based on nonlinear equations of multiphase media dynamics (see, e.g., [1]). In the far-field region, which is the object of study in this paper, the substance concentrations decrease considerably due to turbulent mixing and solid deposition. The suspended substances undergo passive dispersion (see, e.g., [2]) and can be treated as a passive scalar whose transport is determined only by the flow velocity and the intensity of turbulent diffusion. Moreover, the superposition principle applies in the far-field region. This means that the spread of a suspended substance can be represented as the motion of a collection of individual noninteracting "clouds" produced by instantaneous point sources of pollutants. These clouds move through the water under the action of local currents and possibly deposit on the bottom. In the course of motion, they increase in size due to horizontal turbulent diffusion, while the concentrations of suspended substances in them decrease. The suspension concentration C at an arbitrary point $\mathbf{r}$ of the water body is represented as the sum of the passive scalar concentrations C^j in individual clouds including this point at a given time. For example, for a time-continuous stationary point source that releases a polydisperse suspension starting at $t = 0$, we have

$$C(\mathbf{r}, t) = \int_0^t \sum_{j=1}^N C^j(\mathbf{r}, t - t_0, t_0) dt_0, \quad t > 0,$$

where j is the index of a pollutant fraction and N is the number of fractions.

In many practically interesting cases, the three-dimensional numerical simulation of pollutant transport is, at least, unjustified or difficult to perform due to the following factors: the size of the spread area considerably exceeds the depth of the water body; the number of various substance fractions is large; the deposition rates of these fractions may differ by many orders of magnitude; and the control-point concentrations, which have to be reliably computed by the numerical model, differ by four and more orders of magnitude from the suspension concentration at the pollution source.

Two-dimensional (depth-averaged) models are frequently considered in practice. They are based on the integral relation

$$\frac{\partial H\bar{C}^j}{\partial t} + \frac{\partial}{\partial x_i}[H(\overline{u_iC^j} + \bar{J}_i^j)] + J_z^j(H) = 0,$$
$$\bar{C}^j = \frac{1}{H}\int_0^H C^j dz, \quad \bar{J}_i^j = \frac{1}{H}\int_0^H J_i^j dz, \quad \overline{u_iC^j} = \frac{1}{H}\int_0^H u_iC^j dz. \tag{1.1}$$

Here and below, $\mathbf{x} = (x_1, x_2)$ are horizontal Cartesian coordinates; z is a vertical coordinate computed from the water surface to the bottom; $H = H(\mathbf{x})$ is the local depth of the water body; $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2)$ is the horizontal current velocity; J_i^j is the horizontal diffusive suspension flux due to turbulent exchange; $J_z^j(H)$ is the flux of particles toward the bottom; the overbar denotes depth-averaged quantities; and summation is implied over the repeated index $i = 1, 2$.

The total depth-averaged concentration $\bar{C}$ of suspended particles and the suspension mass m per unit area deposited on the bottom are determined by the expressions

$$\bar{C} = \sum_{j=1}^N \bar{C}^j, \quad \frac{\partial m}{\partial t} = \sum_{j=1}^N J_z^j(H).$$

Integral relation (1.1) is an exact consequence of the suspension mass conservation law. In computations, one usually uses the approximate expressions $\overline{u_iC^j} = \bar{u}_i\bar{C}^j$, and $J_z^j(H) = W_j\bar{C}^j$, where W_j is the hydraulic coarseness of the jth fraction (the deposition rate of the fraction in stationary water with no vertical turbulent exchange). Then we have the following two-dimensional "depth-averaged" advection–diffusion equation for suspended substances (see, e.g., [3]):

$$\frac{\partial H\bar{C}^j}{\partial t} + \frac{\partial}{\partial x_i}[H(\bar{u}_i\bar{C}^j + \bar{J}_i^j)] + W_j\bar{C}^j = 0, \quad \bar{u}_i = \frac{1}{H}\int_0^H u_i dz. \tag{1.2}$$

However, this approach is not always acceptable. Specifically, Eq. (1.2) ignores vertical turbulent exchange, which is substantial for particles with small values of W_j. The interaction of deposited particles with the bottom and the effect of a particular vertical location of the pollution source are not taken into account.

In this paper, for an instantaneous point source of suspension, a technique that is free from the indicated shortcomings is proposed and justified for averaging the three-dimensional advection–diffusion equation. The technique is based on the concept of the time-dependent effective hydraulic coarseness of a polydisperse suspension, which is determined by solving only N one-dimensional evolution problems. The depth-averaged suspension concentration is then found by solving the two-dimensional advection–diffusion equation.

We do not consider any solution methods for this equation. Note only that the cloud method and the stochastic discrete particle method (see, e.g., [4–6]) can be used within the approach proposed below.

2. FORMULATION OF THE PROBLEM IN THE CASE OF AN INSTANTANEOUS POINT SOURCE OF SUSPENSION

Assuming that turbulent mixing can be divided into horizontal turbulent exchange and vertical turbulent diffusion (see, e.g., [7]), the three-dimensional equation and the initial condition for determining the far-field evolution of the jth fraction concentration in a suspension produced by an instantaneous point source can be written as

$$\frac{\partial C^j}{\partial t} + \frac{\partial u_i C^j}{\partial x_i} + \frac{\partial (u_z + W_j)C^j}{\partial z} + \frac{\partial J_i^j}{\partial x_i} - \frac{\partial}{\partial z}K_z\frac{\partial C^j}{\partial z} = 0, \tag{2.1}$$

$$C^j = M_j\delta(\mathbf{x})\delta(z - z_0) \quad \text{at} \quad t = 0, \quad C^j \longrightarrow 0 \quad \text{as} \quad |\mathbf{x}| \longrightarrow \infty.$$

Here, u_z is the vertical current velocity, J_i^j are the components of the horizontal turbulent suspension flux, K_z is the vertical turbulent diffusivity, M_j is the initial mass of the suspension, δ is the Dirac delta function, and z_0 is the vertical coordinate of the suspension source. It is assumed that the suspension source is located at the point $\mathbf{x} = 0$ and starts operating at the time $t = 0$.

At the water surface, there is no suspension flux, so

$$W_i C^j - K_z \partial C^j/\partial z = 0 \quad \text{at} \quad z = 0.$$

The boundary condition at the bottom of the reservoir depends on the interaction of the deposited substance with the bottom surface. In the general case, this condition is written as

$$W_j C^j - K_z \partial C^j/\partial z = W_j \beta_j C^j \quad \text{at} \quad z = H(\mathbf{x}), \tag{2.2}$$

where β_j is a parameter that, following the terminology in [8], can be referred to as a dimensionless mass transfer coefficient to the bottom.

The values of β_j depend on the adsorbing properties of the bottom surface. For a completely adsorbing surface, $\beta_j = \infty$ (i.e., the conditions $C^j = 0$ must be set at the bottom instead of (2.2)). No diffusive flux ($\partial C^j/\partial z = 0$) is sometimes specified at the bottom (see [9]), which corresponds to $\beta_j = 1$. The limiting case is $\beta_j = 0$, when there is no suspension flux to the bottom (the bottom is a totally nonadsorbing surface). Note that $\beta_j < 0$ corresponds to the regime of disturbing the deposits at the bottom. It is not considered in this paper.

In what follows, we use the following assumptions.

Assumption 1. The depth of the water body varies relatively slowly:

$$|\partial H/\partial x_i| \ll 1, \quad i = 1, 2.$$

Assumption 2. The shallow water approximation is used, and the vertical turbulent diffusivity K_z in the entire considered domain can be represented as $K_z = u_* H K(\xi)$ with $\xi = z/H$, where the constant u_* is the characteristic vertical diffusion rate, which can sometimes be identified with the friction velocity in the bottom boundary layer; ξ is the dimensionless vertical coordinate; and $K(\xi)$ is the nondimensional vertical turbulent diffusivity.

Assumption 3. The components u_i of the horizontal velocity $\mathbf{u}$ are independent of z; i.e., they are represented as $u_i = \bar{u}_i(\mathbf{x}, t)$.

Assumption 4. The components J_i^j of the horizontal turbulent suspension flux as a function of C^j are given by linear operators independent of z: $J_i^j = \bar{J}_i[C^j]$. We use the hypothesis of gradient horizontal turbulent transfer, which is traditional for the problem in question: $J_i^j = -\bar{A}(\mathbf{x}, t)\partial C^j/\partial x_i$, where $\bar{A}$ is the vertically averaged horizontal turbulent diffusivity. In the case of free horizontal turbulence, however, this hypothesis may be physically unjustified, and the operators $\bar{J}_i[C^j]$, though linear, may have a more complex structure (see [5, 6]).

For a time-dependent current velocity $\mathbf{u}$, Assumption 2, which states that the friction velocity in the boundary layer at the bottom of the reservoir is constant in the entire computational domain, is rather restrictive. This assumption holds, for example, if the characteristic time of varying the horizontal current speed exceeds the time required for scattering a pollution cloud. It can be assumed that this assumption also holds approximately on the oceanic shelf when the current is caused by tidal processes, the eccentricity of the current velocity hodograph is not high, and the current speed does not vary too strongly.

Assumptions 3 and 4 seem relatively severe. However, as a rule, there is no detailed information on the spatial distribution of the current velocity in practice. Moreover, the velocity field $\mathbf{u}$ required for computing suspension transport is frequently obtained by numerical simulation based on the two-dimensional shallow water equations. The corresponding numerical results, which, strictly speaking, are also based on

Assumption 1, reproduce only vertically averaged distributions of the horizontal current velocity in the area of water.

Due to the assumptions made, u_z in (2.1) can be neglected in the first approximation and the problem can be represented in a form similar to (1.1) and (1.2):

$$\frac{\partial H(\mathbf{x})C^j}{\partial t} + \frac{\partial}{\partial x_i}\{H(\mathbf{x})(\bar{u}_i(\mathbf{x}, t)C^j + \bar{J}_i[C^j])\} + u_*\frac{\partial}{\partial \xi}\left[\varepsilon_j C^j - K(\xi)\frac{\partial C^j}{\partial \xi}\right] = 0, \quad \varepsilon_j = \frac{W_j}{u_*},$$

$$C^j = \frac{M_j}{H(0)}\delta(\mathbf{x})\delta(\xi - \xi_0) \quad \text{at} \quad t = 0,$$

$$\varepsilon_j C^j - K(\xi)\frac{\partial C^j}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0 \quad \text{(on the water surface)}, \tag{2.3}$$

$$\varepsilon_j C^j - K(\xi)\frac{\partial C^j}{\partial \xi} = \varepsilon_j \beta_j C^j \quad \text{at} \quad \xi = 1 \quad \text{(on the bottom)}.$$

Here, $H(0)$ is the depth of the water body at the discharge point, ξ_0 is the dimensionless vertical coordinate of the suspension source, and ε_j is a dimensionless parameter equal to the ratio of the particle deposition rate to the characteristic velocity of vertical turbulent diffusion. In practice, the values of this parameter can vary widely. The limit $\varepsilon_j \longrightarrow 0$ corresponds to the case of a fine-particle nondepositing (so-called conservative) suspension. When $\varepsilon_j \longrightarrow \infty$ (large-particle suspension), vertical turbulent mixing can be ignored.

The operator of averaging concentrations over the depth is defined in the usual manner as

$$\bar{C}^j = \int_0^1 C^j(\mathbf{x}, t, \xi)G(\xi)d\xi, \quad \int_0^1 G(\xi)d\xi = 1. \tag{2.4}$$

Here, the nonnegative function $G(\xi)$ is the averaging kernel. Specifically, if $G \equiv 1$, then $\bar{C}^j$ are the depth-averaged concentrations used in (1.1), (1.2).

Below, we formulate the equations and initial conditions for the total concentration $\bar{C} = \Sigma_{j=1}^N \bar{C}^j$ if the function G does not vanish on the interval [0, 1].

3. EXPANSION OVER VERTICAL DIFFUSION MODES

Consider the eigenvalue problem

$$\frac{d}{d\xi}\left[\varepsilon_j Z_n^j - K(\xi)\frac{dZ_n^j}{d\xi}\right] = \lambda_n^j Z_n^j, \tag{3.1a}$$

$$\varepsilon_j Z_n^j - K(0)\frac{dZ_n^j}{d\xi} = 0 \quad \text{at} \quad \xi = 0, \quad \varepsilon_j Z_n^j - K(1)\frac{dZ_n^j}{d\xi} = \varepsilon_j\beta_j Z_n^j \quad \text{at} \quad \xi = 1. \tag{3.1b}$$

By multiplying both sides of Eq. (3.1a) by

$$\rho^j(\xi) = \exp\left[-\varepsilon_j\int_0^\xi K^{-1}(\xi')d\xi'\right],$$

problem (3.1) can be reduced to the classical Sturm–Liouville eigenvalue problem for the equation

$$-\frac{d}{d\xi}\left[p^j(\xi)\frac{dZ_n^j}{d\xi}\right] = \lambda_n^j\rho^j(\xi)Z_n^j, \quad p^j(\xi) = \rho^j(\xi)K(\xi),$$

with boundary conditions (3.1b).

It is well known (see, e.g., [10]) that, for $p^j(\xi) \geq \text{const} > 0$,[1] this problem has a countable set of simple nonnegative eigenvalues $\lambda_n^j = \lambda_n^j(\varepsilon_j, \beta_j)$, $(n = 0, 1, \ldots)$ that depend on the parameters ε_j and β_j (it is

[1] In some cases, the coefficients may have singularities (specifically, vanish) at the endpoints of the interval $0 < \xi < 1$.

assumed below that λ_n^j are arranged in increasing order). Moreover, the eigenfunctions $Z_n^j = Z_n^j(\xi; \varepsilon_j, \beta_j)$, which are referred to hereafter as vertical diffusion modes, form a complete system and satisfy the orthogonality relation

$$(Z_n^j, Z_m^j) = \int_0^1 Z_n^j(\xi) Z_m^j(\xi) \rho^j(\xi) d\xi = \delta_{nm}, \tag{3.2}$$

where δ_{nm} is the Kronecker delta.

A solution to problem (2.3) is sought in the form of the series

$$C^j = \sum_{n=0}^{\infty} C_n^j(\mathbf{x}, t) Z_n^j(\xi), \quad C_n^j = (C^j, Z_n^j). \tag{3.3}$$

Substituting (3.3) into (2.3) and using orthogonality relation (3.2), in view of (3.1a), we easily obtain equations for the expansion coefficients C_n^j:

$$\begin{aligned} &\frac{\partial H C_n^j}{\partial t} + \frac{\partial}{\partial x_i}\{H(\bar{u}_i C_n^j + \bar{J}_i[C_n^j])\} + u_* \lambda_n^j C_n^j = 0, \\ &C_n^j = \frac{M_j}{H(0)} \delta(\mathbf{x}) Z_n^j(\xi_0) \rho^j(\xi_0) \quad \text{at} \quad t = 0. \end{aligned} \tag{3.4}$$

According to (3.3), the depth-averaged total concentration distribution $\bar{C}$ can be found as follows:

$$\bar{C} = \sum_{j=1}^{N} \sum_{n=0}^{\infty} C_n^j(\mathbf{x}, t) \bar{Z}_n^j, \quad \bar{Z}_n^j = \int_0^1 Z_n^j(\xi; \varepsilon_j, \beta_j) G(\xi) d\xi. \tag{3.5}$$

4. LONG-TIME ASYMPTOTICS

It can be seen that the solutions to problems (3.4) tend exponentially to zero as $t \longrightarrow \infty$. The slowest decaying components C_0^j in (3.3) correspond to the minimum eigenvalues λ_0^j, which can be called the dimensionless hydraulic coarseness of fraction j with allowance for vertical turbulent exchange (see (1.2)). In other words, as $t \longrightarrow \infty$, the leading term in (3.5) is that with the fraction index $j = j_m$, for which λ_0^j is minimal. Note that the index j_m not necessarily corresponds to the fraction with the minimum hydraulic coarseness (i.e., with the minimum value of ε_j) since λ_0^j also depends on β_j, which takes into account the interaction of the depositing suspension with the bottom.

In the general case, the vertical diffusivity $K(\xi)$ has an inhomogeneous distribution over depth and eigenvalue problem (3.1) can be solved only by numerical methods. However, when $K = 1$, the problem has the analytical solution

$$\lambda_n^j = \frac{\varepsilon_j^2}{4} + (\omega_n^j)^2, \quad Z_n^j(\xi) = \frac{\varepsilon_j \sin(\omega_n^j \xi) + 2\omega_n^j \cos(\omega_n^j \xi)}{(D_n^j)^{1/2}} \exp\left(\frac{\varepsilon_j \xi}{2}\right),$$

$$D_n^j = \frac{\varepsilon_j^2}{2} + 2(\omega_n^j)^2 + \frac{2}{\omega_n^j}\left[(\omega_n^j)^2 - \frac{\varepsilon_j^2}{4} \sin\omega_n^j \cos\omega_n^j\right] + 2\varepsilon_j^2 \sin^2\omega_n^j.$$

Here, ω_n^j are the positive roots of the equation

$$\frac{4\beta_j \omega_n^j \varepsilon_j}{4(\omega_n^j)^2 - (2\beta_j - 1)\varepsilon_j^2} = \tan\omega_n^j.$$

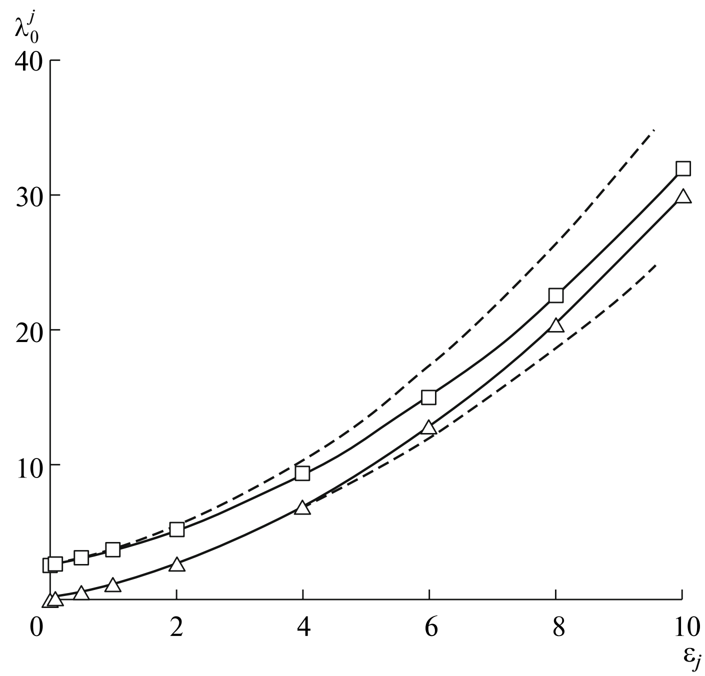

Fig. 1.

For $\beta_j > 0$, this equation has a countable set of positive roots separated roughly by π with the minimum positive root ω_0^j lying in the range $0 < \omega_0^j < \pi$.

For fixed β_j and $\varepsilon_j \longrightarrow 0$, the principal eigenvalue λ_0^j has the asymptotic expansion

$$\lambda_0^j = \beta_j\varepsilon_j + \frac{1}{6}\beta_j(3 - 2\beta_j)\varepsilon_j^2 + O(\varepsilon_j^3), \tag{4.1}$$

which is not regular as $\beta_j \longrightarrow \infty$. Without performing a detailed analysis, we indicate that, for $\beta_j = \infty$ (a completely adsorbing bottom surface) and $\varepsilon_j \longrightarrow 0$, we have

$$\lambda_0^j = \left(\frac{\pi}{2}\right)^2 + \varepsilon_j + \frac{\varepsilon_j^2}{4} + O(\varepsilon_j^3). \tag{4.2}$$

Figure 1 shows the plots of $\lambda_0^j(\varepsilon_j)$ for $\beta_j = 1$ (curve with triangles) and $\beta_j = \infty$ (curve with squares) and asymptotics (4.1) and (4.2) (dashed curves).

In view of the above analysis, in some cases, the shape of the pollution plume at a sufficiently long time after a pollutant discharge can be found by the numerical integration of the two-dimensional advection–diffusion equation (3.4) for the fraction that minimizes λ_0^j. However, if we need to compute the total amount of suspension deposited on the bottom and the suspension concentrations over the entire time interval of plume existence, then series (3.3) and (3.5) have to be summed. An analysis of the case of $K = 1$ shows that, for $\varepsilon_j < 1$, series (3.3) and the inner series in (3.5) are well summable, so, for small ε_j, it is sufficient to retain only the principal diffusion mode Z_0^j in these expansions. However, the convergence of the series degrades tremendously with increasing ε_j and decreasing t. This is not surprising, since problem (2.3) degenerates in the limit as $\varepsilon_j \longrightarrow \infty$. Fortunately, there is a technique for overcoming this difficulty.

5. EQUATION FOR THE DEPTH-AVERAGED CONCENTRATION OF SUSPENDED SUBSTANCES IN A CONSTANT-DEPTH WATER BODY

Applying summation operator (3.5) to Eqs. (3.4) yields the following relations for the depth-averaged suspension concentration $\bar{C}$:[2]

$$\frac{\partial H\bar{C}}{\partial t} + \frac{\partial}{\partial x_i}\{H(\bar{u}_i\bar{C} + \bar{J}_i[\bar{C}])\} + W\bar{C} = 0, \quad \bar{C} = \frac{M}{H(0)}\delta(\mathbf{x})G(\xi_0) \quad \text{при} \quad t = 0, \tag{5.1a}$$

$$W = u_* w, \quad w = \sum_{j=1}^{N}\sum_{n=0}^{\infty}\lambda_n^j C_n^j \bar{Z}_n^j \Big/ \sum_{j=1}^{N}\sum_{n=0}^{\infty} C_n^j \bar{Z}_n^j, \quad M = \sum_j M_j. \tag{5.1b}$$

Here, M is the total initial mass of pollutants and W can be called the effective hydraulic coarseness of the suspension (w is the dimensionless effective hydraulic coarseness).

For a constant-depth water body with $H = \text{const}$, we show that w is independent of $\mathbf{x}$ but is a function of t and describe a technique for determining w.

In the case under study, it is easy to see that the solutions to problems (3.4) can be represented as

$$C_n^j = \frac{M_j}{H} Z_n^j(\xi_0)\rho^j(\xi_0)\mu^0(\mathbf{x}, t)\exp(-u_*\lambda_n^j t/H). \tag{5.2}$$

Here, the function C^0 independent of j or n describes the conservative dispersion of a cloud of unit mass and satisfies the equation

$$\frac{\partial \mu^0}{\partial t} + \frac{\partial}{\partial x_i}(\bar{u}_i\mu^0 + \bar{J}_i[\mu^0]) = 0, \quad \mu^0 = \delta(\mathbf{x}) \quad \text{at} \quad t = 0. \tag{5.3}$$

Substituting (5.2) into (5.1b) gives

$$w \equiv w(t) = \frac{\sum_{j=1}^{N} M_j \sum_{n=0}^{\infty}\lambda_n^j Z_n^j(\xi_0)\rho^j(\xi_0)\exp(-u_*\lambda_n^j t/H)\bar{Z}_n^j}{\sum_{j=1}^{N} M_j \sum_{n=0}^{\infty} Z_n^j(\xi_0)\rho^j(\xi_0)\exp(-u_*\lambda_n^j t/H)\bar{Z}_n^j}. \tag{5.4}$$

As was mentioned above, for arbitrary ε_j and t, the inner series in (5.4) are poorly summable and attempts to determine $w(t)$ by solving eigenvalue problem (3.1) and applying summation (5.4) fail. However, this function is independent of the current velocity $\mathbf{u}$ or the horizontal diffusive flux components $\bar{J}_i$. Therefore, it can be found by integrating the following collection of one-dimensional evolution problems (see (2.3)), averaging their solutions over ξ, summing up the results over the fraction index j, and computing the logarithmic derivative:

$$\frac{\partial Hc^j}{\partial t} + u_*\frac{\partial}{\partial \xi}\left[\varepsilon_j c^j - K(\xi)\frac{\partial c^j}{\partial \xi}\right] = 0, \quad c^j = \frac{M_j}{H}\delta(\xi - \xi_0) \quad \text{at} \quad t = 0,$$

$$\varepsilon_j c^j - K(\xi)\frac{\partial c^j}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0, \quad \varepsilon_j c^j - K(\xi)\frac{\partial c^j}{\partial \xi} = \varepsilon_j\beta_j c^j \quad \text{at} \quad \xi = 1, \tag{5.5}$$

$$\bar{c} = \sum_{j=1}^{N}\int_0^1 c^j G(\xi)d\xi, \quad w(t) = -\frac{H}{u_*}\frac{d\ln\bar{c}}{dt}.$$

It is easy to see that w thus determined depends on the initial disperse composition of the suspension determined by ε_j and M_j, on the nondimensional mass release coefficient β_j, and on the parameter ξ_0 specifying the location of the suspension source over the bottom. Moreover, w also depends on the concentration-averaging method in the vertical direction (i.e., on the chosen form of $G(\xi)$).

Thus, for a constant-depth water body, the computation of the vertically averaged dispersion of a polydisperse suspension with N fractions is reduced to solving N one-dimensional evolution problems (5.5)

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[2] The initial condition for Eq. (5.1a) was derived using the delta function expansion $\delta(\xi - \xi_0) = \sum_{n=0}^{\infty} Z_n^j(\xi_0)\rho^j(\xi_0)Z_n^j(\xi)$.

and, then, one two-dimensional evolution problem (5.1a). The solution of the latter is given by the formula

$$\bar{C} = \frac{M}{H}G(\xi_0)\mu^0(\mathbf{x}, t)\exp\left(-\frac{u_*}{H}\int_0^t w(t')dt'\right).$$

Importantly, for a time-continuous and/or spatially distributed source with a constant disperse composition, evolution problems (5.5) can be solved only once.

6. TRAJECTORY APPROXIMATION FOR THE CASE OF A VARIABLE-DEPTH WATER BODY

In the general case, when the depth H is not a constant, the solutions to problems (3.4) cannot be represented in the form of (5.2). Therefore, strictly speaking, the above approach does not apply. However, we can propose an approximate method associated with the concept of the trajectory of a suspension cloud. This method is applicable assuming that the depth of the water body varies little at distances on the order of the characteristic length of the evolving cloud.

The trajectory of motion of the cloud center $\mathbf{x}_0(t)$ is defined in the usual manner as $d\mathbf{x}_0/dt = \mathbf{u}(\mathbf{x}_0, t)$, $\mathbf{x}_0(0) = 0$. Let $H_0(t) = H(\mathbf{x}_0(t))$ be the time-varying depth of the water body at the center of the cloud.

Under the assumption formulated above, the solutions to problems (3.4) can be approximately represented as

$$C_n^j = \frac{M_j}{H}Z_n^j(\xi_0)\rho^j(\xi_0)\mu^0(\mathbf{x}, t)\exp\left(-u_*\lambda_n^j\int_0^t H_0^{-1}(t')dt'\right), \tag{6.1}$$

where μ^0 is a compactly supported function satisfying Eq. (5.3).

Indeed, substituting (6.1) into (3.4), we obtain

$$\frac{\partial\mu^0}{\partial t} + \frac{\partial}{\partial x_i}(\bar{u}_i\mu^0 + \bar{J}_i[\mu^0]) = u_*\lambda_n^j\left(\frac{1}{H_0(t)} - \frac{1}{H(\mathbf{x})}\right)\mu^0. \tag{6.2}$$

Let the characteristic half-width of μ^0 be $\sigma(t)$ and be small in comparison with the characteristic length scale of variations in the reservoir depth. Then the right-hand side of (6.2) is estimated by the small value

$$\frac{\sigma^2(t)}{H_0(t)}\max_i\left|\partial H/\partial x_i\right|_{\mathbf{x} = \mathbf{x}_0(t)}\Big| \ll 1.$$

If approximate representation (6.1) can be used, then all the arguments presented in Section 5 are valid. Specifically, we have Eqs. (5.1a) and (5.5) with H replaced by $H_0(t)$.

In this case, $w(t)$ becomes depending on the trajectory of the suspension cloud. However, as before, for a time-continuous and/or spatially distributed source, evolution problems (5.5) are solved only once. Indeed, making in (5.5) the substitutions

$$\mu^j = H_0(t)c^j, \quad \tau = u_*\int_0^t H_0^{-1}(t')dt',$$

we obtain

$$\frac{\partial\mu^j}{\partial\tau} + \frac{\partial}{\partial\xi}\left[\varepsilon_j\mu^j - K(\xi)\frac{\partial\mu^j}{\partial\xi}\right] = 0, \quad \mu^j = M_j\delta(\xi - \xi_0) \quad \text{at} \quad t = 0,$$

$$\varepsilon_j\mu^j - K(\xi)\frac{\partial\mu^j}{\partial\xi} = 0 \quad \text{at} \quad \xi = 0, \quad \varepsilon_j\mu^j - K(\xi)\frac{\partial\mu^j}{\partial\xi} = \varepsilon_j\beta_j\mu^j \quad \text{at} \quad \xi = 1, \tag{6.3}$$

$$\bar{\mu} = \sum_{j=1}^N\int_0^1\mu^j G(\xi)d\xi, \quad w(\tau) = -\frac{d\ln\bar{\mu}}{d\tau}.$$

The standard function $w(\tau)$ thus defined is independent of the trajectory of the suspension cloud.

Table

Fraction index j	Particle size range, mm	Mean diameter D_j, mm	Mass fraction in discharge M_j, %	Hydraulic coarseness W_j, m/s	ε_j
	>40		0.00		
1	40–20	30	1.65	9.845E–01	1.969E+01
2	20–10	15	1.28	6.955E–01	1.391E+01
3	10.0–5.0	7.5	2.17	4.906E–01	9.812E+00
4	5.0–2.0	3.5	2.28	3.323E–01	6.647E+00
5	2.0–1.0	1.5	2.56	2.109E–01	4.217E+00
6	1.0–0.5	0.75	4.58	1.377E–01	2.754E+00
7	0.5–0.25	0.375	8.87	7.823E–02	1.565E+00
8	0.25–0.1	0.175	10.25	2.921E–02	5.841E–01
9	0.1–0.05	0.075	11.56	6.216E–03	1.243E–01
10	0.05–0.01	0.03	21.25	1.010E–03	2.019E–02
11	0.01–0.005	0.0075	15.10	6.316E–05	1.263E–03
12	<0.005		18.45	0.000E+00	0.000E+00

To conclude, we note that, when vertical turbulent mixing can be neglected, W in Eq. (5.1a) must be calculated by the formula

$$W(t) = \sum_{j=1}^{N} M_j W_j \exp\left(-W_j \int_0^t H_0^{-1}(t')dt'\right)\left[\sum_{j=1}^{N} M_j \exp\left(-W_j \int_0^t H_0^{-1}(t')dt'\right)\right]^{-1}.$$

7. EXAMPLES OF COMPUTING THE EFFECTIVE HYDRAULIC COARSENESS OF A POLYDISPERSE SUSPENSION

As an illustration, we computed the effective hydraulic coarseness of an actual polydisperse suspension, namely, light sandy loam, which is met in dredging operations. Its initial disperse composition, the

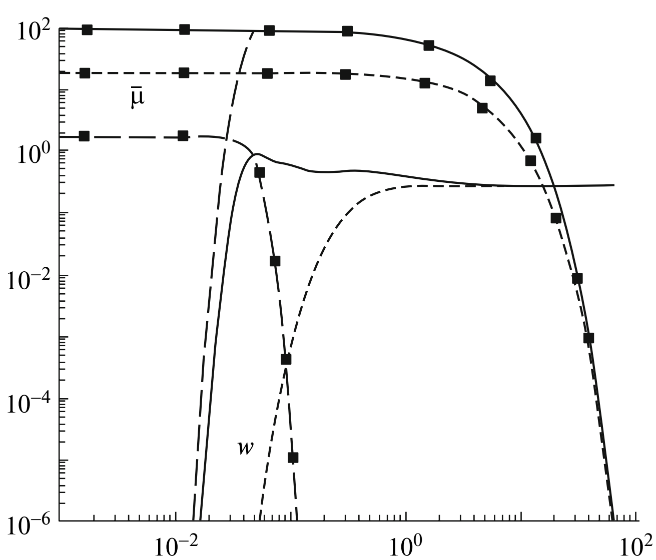

Fig. 2.

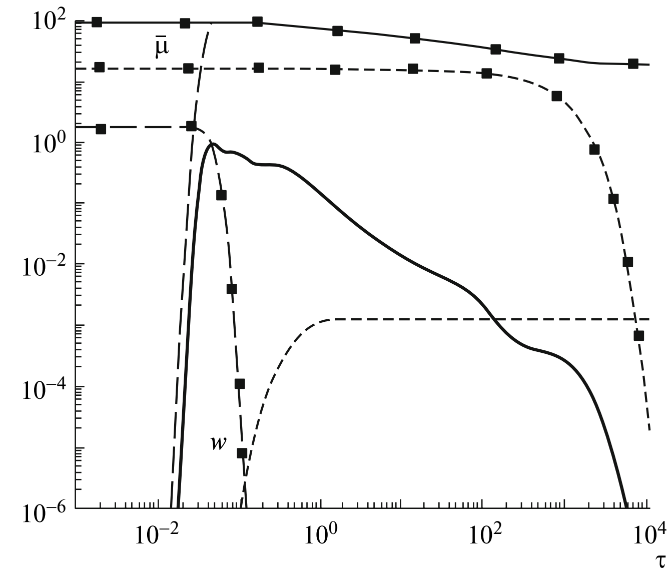

Fig. 3.

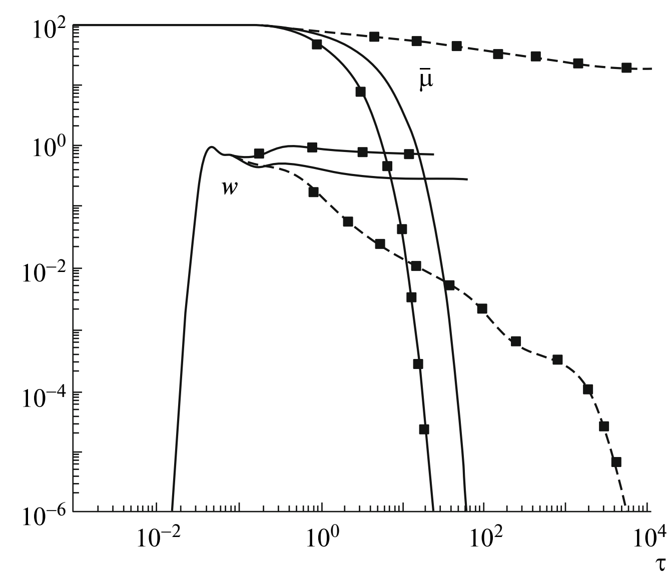

Fig. 4.

hydraulic coarseness of the components, and the values of ε_j are presented in the table. To compute ε_j, the friction velocity at the bottom was specified as $u_* = 0.05$ m/s. The dispersion of the finest fraction with the particle diameter $D_j < 0.005$ mm was treated as conservative; i.e., we set $W_j = 0$.

The dimensionless profile of the vertical turbulent diffusivity was specified by the following expression (see, e.g., [11, 12])[3]:

$$K(\xi) = (1 - \xi + \delta)(0.4 + 0.6\xi). \tag{7.1}$$

Here, δ is a small parameter equal to the ratio of the dimensionless bottom roughness to the depth.

[3] Note that the vertical coordinate z in [11, 12] was measured from the bottom to the reservoir surface.

In the computations, we set $\delta = 0.01$. The dimensionless vertical coordinate of the suspension source was $\xi_0 = 0.1$. We determined the effective hydraulic coarseness of the suspension corresponding to the vertically averaged concentration of suspended substances ($G(\xi) = 1$).

Problem (6.3) was numerically integrated using an implicit conservative difference scheme, which was solved by tridiagonal Gaussian elimination. The computations were performed for the following two cases:

(i) The bottom is completely adsorbing; i.e., $\beta_j = \infty$ for all the suspension fractions (see Fig. 2).

(ii) There are no diffusive fluxes at the bottom; i.e., $\beta_j = 1$ for all the fractions (see Fig. 3).

Both figures show the calculated dynamics of $\bar{\mu}(\tau)$ (curves with squares) and $w(\tau)$. The solid curves correspond to the polydisperse suspension presented in the table. The dashed curves depict the results obtained for the monodisperse suspension corresponding to the largest fraction in the table ($j = 1$). The dotted curves correspond to a monodisperse conservative suspension ($j = 12$ in the table) in Fig. 2. In Fig. 3, they depict the results for the monodisperse suspension corresponding to the finest nonconservative fraction ($j = 11$ in the table).

It can be seen that, at the initial times τ, the mean mass $\bar{\mu}(\tau)$ of suspended particles does not vary, since some time is required for a compact suspension cloud to achieve the bottom of the water body. Therefore, $w(\tau) = 0$. The subsequent evolution of $\bar{\mu}(\tau)$ and $w(\tau)$ is associated with the consecutive deposition of various suspension fractions on the bottom. In the case of a completely adsorbing bottom (see Fig. 2) as $\tau \longrightarrow \infty$, the evolution of these functions is determined by the conservative suspension component adsorbed by the bottom. In the case of no diffusive fluxes to the bottom (Fig. 3), the conservative component is not adsorbed and this fraction always remains in a suspended state, so that $w(\tau) = 0$ for sufficiently long times τ.

For test purposes, we also considered the case of a constant vertical turbulent diffusivity

$$\bar{K} = \int_0^1 K(\xi)d\xi = 0.3 + 0.7\delta \approx 0.3. \tag{7.2}$$

The numerical results are presented in Fig. 4. The solid curves correspond to $\beta_j = \infty$, and the dashed curves, to $\beta_j = 1$. The case of the vertical diffusivity given by (7.1) is shown by curves without markers, while the curves with markers correspond to the averaged vertical diffusivity (7.2). For $\beta_j = 1$, the curves nearly coincide. As was expected, noticeable differences are observed only when the evolution of $\bar{\mu}(\tau)$ and $w(\tau)$ is determined by the diffusion of the suspension to the bottom ($\beta_j = \infty$). In the case of no diffusive fluxes, the curves coincide, which can be explained as follows: on the one hand, vertical diffusion is negligible for large fractions; on the other hand, in the case of fine fractions at $\beta_j = 1$, this process ensures intense turbulent mixing of the suspension regardless of the vertical diffusivity profile.

8. CONCLUSIONS

It was shown that, even if vertical turbulent exchange has a large effect, the problem of computing the evolution of polydisperse suspended pollutants produced by an instantaneous point source in shallow water bodies can be reduced to the integration of N one-dimensional evolution problems (N is the number of suspension fractions) and to the solution of one two-dimensional problem. This result can be used to design efficient numerical methods intended for the mathematical modeling of the dispersion of a suspension produced by a time-continuous and/or spatially distributed source of water pollution.

The concept of effective hydraulic coarseness, which takes into account the adsorption properties of the bottom, can be used to compute pollutant dispersion in rivers, where the flow is frequently simulated using the one-dimensional Saint-Venant equations (see, e.g., [13]). The approach described can be useful for the development of prognostic mathematical models for urban air quality dynamics based on the advection–diffusion equation when detailed three-dimensional simulation (see, e.g., [14, 15]) is impossible.

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