On the Summation of Fourier Series over the Roots of Transcendental Equations Using Annihilation

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Introduction

- We consider the problem of symbolic summation of trigonometric series containing roots of a transcendental equation instead of natural numbers of harmonics.
- We present a continuation of our previous work on ordinary Fourier series. This work was recently published. [Malyshev, Malykh, Sevastyanov, Zorin, Mathematics, 2025].

Problem Formulation

Today we consider series of the type

$$u = \sum_{\lambda} \frac{P}{Q} \sin \lambda x, x \in [0,1], \qquad P \in \mathbb{R}[\lambda], Q \in \mathbb{R}[\lambda].$$

Summation is carried out over the positive roots of the equation

$$\lambda \cos \lambda + h \sin \lambda = 0, \qquad h > 0.$$

Expressions for P and Q as functions of lambda are given. It is required to determine the expression for u in its final form.

Motivation and Idea

These series can be considered as Fourier series in eigenfunctions of the Sturm-Liouville problem:

$$y'' + \lambda^2 y = 0, \quad x \in (0,1)$$

$$y(0) = 0, \quad y'(1) + hy(1) = 0, h > 0.$$

Eigenfunctions:

$$y_\lambda = \sin \lambda x$$

Square of the norm of an eigenfunction:

$$\|y_\lambda\|^2_{L_2(0,1)} = \frac{1}{2}\frac{h^2 + h + \lambda^2}{h^2 + \lambda^2}$$

Let's consider the summation:

$$u = \sum_\lambda a_\lambda \frac{h^2 + \lambda^2}{h^2 + h + \lambda^2} \sin \lambda x, \qquad x \in [0,1], \qquad a_\lambda \in \mathbb{R}(\lambda).$$

Usual Fourier Series: $h = \infty$

Usual case: $\lambda \in \mathbb{N}$. This occurs when replacing $[0, 1]$ with $[0, \pi]$ and formally replacing the final value $h > 0$ with $h = \infty$.

$$u = \sum_{n=1}^{\infty} a_n \sin nx, \qquad x \in [0, \pi], \qquad a_n \in \mathbb{R}(n).$$

Let us show how such a series can be summed, following our previous work [Malyshev at all, Mathematics, 2025].

Annihilation usual series ($h = \infty$). Step 1.

Theorem

Let $a_n \in \mathbb{R}(n)$. Then there exists a linear differential expression $L \in \mathbb{R}[\frac{d^2}{dx^2}]$ and a polynomial $A_n \in \mathbb{R}[n]$, that the sum of the series

$$u = \sum_{n=1}^{\infty} a_n \sin nx$$

is a solution in the sense of $\mathfrak{D}'$ of the linear differential equation

$$L[u] = \sum_{n=1}^{\infty} A_n \sin nx, \qquad A_n \in \mathbb{R}[n]. \tag{1}$$

Let us introduce the notation L_Q for the operator $L \in \mathbb{R}[D^2]$ that annihilates the denominator $Q_n \in \mathbb{R}[n]$.

Basic Divergent Series for $h = \infty$

Note that

Lemma

In the space $\mathfrak{D}'$ the following formulas are valid

$$\sum_{n=1}^{\infty} \widehat{\cos nx} = \pi\widehat{\delta(x,0)} - \frac{\hat{1}}{2}$$

and

$$\sum_{n=1}^{\infty} \widehat{\sin nx} = \frac{1}{2}\widehat{\cot \frac{x}{2}}.$$

Annihilation usual series ($h = \infty$). Step 2.

- Note that the distribution $\sum_{n=1}^{\infty} B_n \sin nx, \qquad B_n \in \mathbb{R}[n]$ is a finite sum of a constant, a δ-function, a cot, and their derivatives, denoted by f.
- It is necessary to find a solution to the equation $L[u] = f$ in the space $\mathfrak{D}'$.
- If f does not contain $\widehat{\cot}$, the solution is easy to find. This is the so-called case of A.N. Krylov series.

On the solution of the differential equation in A.N. Krylov case

Theorem

Let the operator L have order q"— the largest order of differentiation in the expression L. Let

$$f = \alpha + \alpha_0\delta(x) + \sum_{n=1}^{q-1} \alpha_n D^n \delta(x), \qquad \alpha \in \mathbb{R}, \alpha_n \in \mathbb{R}.$$

Let the operator L have no zero eigenvalue under the periodicity boundary conditions. Then there exists a unique function u satisfying the equation $L[u] = f$ in the sense of $\mathfrak{D}'$. This function is expressed in the finite form of piecewise elementary functions of the variable x, this expression can be found in a finite number of steps.

Example of A.N. Krylov's series

Example

$$u = \sum_{n=1}^{\infty} \frac{n^3}{1+n^4} \sin nx$$

Cf. [Krylov], [Pak], [Tolstov].

$$L[u] = (D^4+1)u = \pi D^3 \delta(x), \qquad x \in [-\pi, \pi];$$

$$u(-\pi) = u(\pi) = 0, \qquad D^2 u(-\pi) = D^2 u(-\pi) = 0.$$

The solution to the problem is the result of the summation:

$$u = \frac{\pi}{2} \frac{\cosh \frac{x}{\sqrt{2}} \cos \frac{2\pi + x}{\sqrt{2}} - \cos \frac{x}{\sqrt{2}} \cosh \frac{2\pi + x}{\sqrt{2}}}{\cosh \pi\sqrt{2} - \cos \pi\sqrt{2}} + \pi \mathfrak{h}(x) \cosh \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}}$$

Fourier series with parameter $0 < h < \infty$

The summation over the positive roots of equation $\lambda \cos \lambda + h \sin \lambda = 0$:

$$u = \sum_{\lambda} a_{\lambda} \frac{h^2 + \lambda^2}{h^2 + h + \lambda^2} \sin \lambda x, \qquad x \in [0,1], \qquad a_{\lambda} \in \mathbb{R}(\lambda).$$

Basic divergent series:

Theorem

In the sense of the space $D'(-1,1)$ the following equalities hold:

$$\sum_{\lambda} \lambda^{2k+1} \frac{h^2 + \lambda^2}{h^2 + h + \lambda^2} \sin \lambda x = 2 \cdot (-1)^{k+1} \cdot D^{2k+1} \delta(x), k \in \mathbb{N} \cup \{0\}$$

Annihilation with parametr $0 < h < \infty$. Step 1.

$$u = \sum_{\lambda} a_\lambda \frac{h^2+\lambda^2}{h^2+h+\lambda^2} \sin \lambda x, \qquad x \in [0,1], \qquad a_\lambda = \frac{P_\lambda}{Q_\lambda} \in \mathbb{R}(\lambda).$$

Let $Q_\lambda \in \mathbb{R}[\lambda^2]$:

$$Q_\lambda = q_N \lambda^{2N} + q_{N-1}\lambda^{2(N-1)} + \ldots + q_0.$$

Then the operator $L_Q \in \mathbb{R}[D^2]$ has the form:

$$L_Q = (-1)^N q_N D^{2N} + (-1)^{N-1} q_{N-1} D^{2(N-1)} + \ldots + q_0.$$

$$L_Q u = \sum_{n=1}^{\infty} P_\lambda \frac{h^2+\lambda^2}{h^2+h+\lambda^2} \sin \lambda x$$

Annihilation with parametr $0 < h < \infty$. Step 2.

$$L_Q u = \sum_{n=1}^{\infty} P_\lambda \frac{h^2 + \lambda^2}{h^2 + h + \lambda^2} \sin \lambda x$$

Let $P_\lambda \in \mathbb{R}[\lambda]$ is an odd function λ. Then

$$\sum_{n=1}^{\infty} P_\lambda \frac{h^2 + \lambda^2}{h^2 + h + \lambda^2} \sin \lambda x = \beta_1 \cdot D^1 \delta(x) + \beta_3 \cdot D^3 \delta(x) + \ldots + \beta_d \cdot D^d \delta(x)$$

$$d = degP_\lambda.$$

It remains to find L^{-1}. Boundary conditions of the third kind:

$$u'|_{x=1} + hu|_{x=1} = 0, u'|_{x=-1} - hu|_{x=-1} = 0,$$

u''' $|_{x=1} + hu''|_{x=1} = 0, u'''|_{x=-1} - hu''|_{x=-1} = 0, etc.$

Annihilation with parametr $0 < h < \infty$

Theorem

Let the annihilation operator L_Q have no zero eigenvalue under boundary conditions of the third kind. Let the polynomial P_λ contain only odd powers of λ. Then the series u is expressed in the finite form of piecewise elementary functions on the segment $[-1, 1]$.

N.S. Koshliakov Polynomials for $0 < h < \infty$

A generalization of Bernoulli polynomials proposed by N.S.Koshliakov in 1935.

$$\sum_{\lambda} \frac{h^2+\lambda^2}{h^2+h+\lambda^2} \frac{\sin \lambda x}{\lambda^{2k+1}} = K_{2k+1}.$$

$$-2K_7 = \frac{x^6}{360}\mathfrak{h}(x) -$$

$$-\frac{h}{5040(h+1)}x^7 - \frac{x^6}{720} - \frac{h^2+3h+3}{360(h+1)^2}x^5 +$$

$$+\frac{h^3+6h^2+15h+15}{270(h^2+2h+1)(h+1)}x^3 - \frac{2h^4+18h^3+72h^2+147h+126}{945(h^3+3h^2+3h+1)(h+1)}x,$$

$$x \in [-1, 1].$$

Example, $h = 1$

Example

Series:

$$u = \sum_{\lambda} \frac{\lambda^5 + \lambda^3}{\lambda^6 + 2\lambda^4 + \lambda^2 + 2} \sin \lambda x$$

Annihilation:

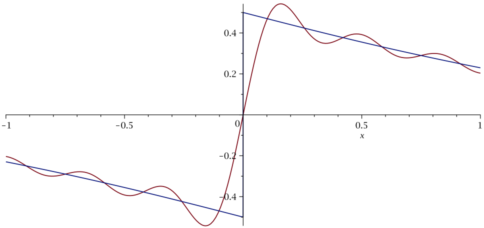
$$L[u] = (D^4 + 1)u = D^3\delta(x), \qquad x \in [-1, 1];$$

$$(u' + u)|_{x=1} = (u' - u)|_{x=-1} = 0 = (u''' + u'')|_{x=1} = (u''' - u'')|_{x=-1}.$$

$$u = c_1 e^{-\frac{x}{\sqrt{2}}} \cos \frac{x}{\sqrt{2}} + c_2 e^{-\frac{x}{\sqrt{2}}} \sin \frac{x}{\sqrt{2}} + c_3 e^{\frac{x}{\sqrt{2}}} \cos \frac{x}{\sqrt{2}} + c_4 e^{\frac{x}{\sqrt{2}}} \sin \frac{x}{\sqrt{2}} +$$

$$+ \mathfrak{h}(x) \cosh \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}}$$

Example, $h = 1$



The sum of the first six terms of the series

$$u = \sum_{\lambda} \frac{\lambda^5 + \lambda^3}{\lambda^6 + 2\lambda^4 + \lambda^2 + 2} \sin \lambda x$$

and the final expression for the sum of the series.

We don't know

$$u = \sum_{\lambda} \frac{h^2+\lambda^2}{h^2+h+\lambda^2} \sin \lambda x = ??$$

In the sense of generalized functions, one can obtain equalities:

$$u|_{h=\infty} = \frac{1}{2} \widehat{\cot \frac{x}{2}}$$

$$u|_{h=0} = \frac{1}{2} \widehat{\frac{1}{\sin x}}$$

What happens for an arbitrary finite value of h? Having defined the sum of this series in a convenient finite form, we can hope to be able to sum even more series.

Conclusions

- A generalization of summation using annihilation is obtained for the case of Fourier series over the roots of a transcendental equation.
- The proposed scheme can be transferred to other variants of trigonometric series with roots of transcendental equations, cf. [Repnikov, Khukhryansky, Bardakov].
- Some questions remain unclear, perhaps other methods of summation are required for them.

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