The TruncatedSeries Package

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Description

The TruncatedSeries package contains commands to find **Laurent** series, formal **regular** and formal **exponential-logarithmic** solutions of homogeneous linear ordinary differential equations with the coefficients which are either algorithmically represented power series, or truncated power series.
The Maple library with the TruncatedSeries package and Maple worksheets with examples of using its procedures are available at the address

http://www.ccas.ru/ca/truncatedseries

The key possibilities of the package are presented here.

Calling Sequence

LaurentSolution(ode, var, opts)

RegularSolution(ode, var, opts)

FormalSolution(ode, var, opts)

Parameters

ode	-	homogeneous linear ODE
var	-	dependent variable, for example y(x)
opts	-	optional arguments of the form keyword=value

The equation ode

The equation for $y(x)$ may be given in the diff-form

$$a_r(x)\left(\frac{d^r}{dx^r}y(x)\right)+\ldots+a_1(x)\left(\frac{d}{dx}y(x)\right)+a_0(x)\,y(x)=0$$

or in the theta-form

$$a_r(x)\,\theta(y(x),x,r)+\ldots+a_1(x)\,\theta(y(x),x,1)+a_0(x)\,y(x)=0$$

where r is a positive integer and

$$\theta(y(x),x,1)=x\frac{d}{dx}y(x)$$

Coefficients $a_r(x)$, ..., $a_1(x)$, $a_0(x)$ of the ode

Coefficients $a_r(x)$, ..., $a_1(x)$, $a_0(x)$ may be

- algorithmically represented power series:
  - a polynomial in x over the algebraic number field;
  - an infinite power sum specified by means of $\sum_{k=k_0}^{\infty} f(k)\, x^k$ with a non negative integer initial value k_0 of the summation index k; the coefficient $f(k)$ of x^k may be given by an arbitrary expression of the index k which gives an algebraic number for all $k \geq k_0$;
  - a sum of a polynomial and an infinite power sum described above;

- truncated power series (t is called **the truncation degree**):
  - $\mathrm{O}(x^{t+1})$ where t is an integer, $t \geq -1$;
  - $a(x) + \mathrm{O}(x^{t+1})$ where $a(x)$ is a polynomial in x over the algebraic number field and t is an integer greater than or equal to the degree of $a(x)$.

Equation examples

An equation whose coefficients are all algorithmically represented power series:

> $f := \mathbf{proc}(i)\ \mathbf{if}\ i :: 'integer'\ \mathbf{then}\ 0\ \mathbf{else}\ 'procname'(i)\ \mathbf{end\ if\ end\ proc}$:

$$x^9 \frac{d^5}{dx^5} y(x) + \left(x^7 + \sum_{k=9}^{\infty} \frac{k^2}{2} x^k\right) \frac{d^4}{dx^4} y(x) + (x^2 + 2x^5) \frac{d^2}{dx^2} y(x) + (3x + x^4 + 2x^{10}) \frac{d}{dx} y(x) + \left(\sum_{k=0}^{\infty} f(k)\, x^k\right) y(x) = 0;$$

$$x^9 \left(\frac{d^5}{dx^5} y(x)\right) + \left(x^7 + \left(\sum_{k=9}^{\infty} \frac{k^2 x^k}{2}\right)\right) \left(\frac{d^4}{dx^4} y(x)\right) + (2x^5 + x^2) \left(\frac{d^2}{dx^2} y(x)\right) + (2x^{10} + x^4 + 3x) \left(\frac{d}{dx} y(x)\right) + \left(\sum_{k=0}^{\infty} f(k)\, x^k\right) y(x) = 0 \qquad (1)$$

An equation whose coefficients are all truncated power series with various truncation degrees:

> $$\mathrm{O}(x^9) \frac{d^5}{dx^5} y(x) + \left(x^7 + \frac{81 x^9}{2} + 50 x^{10} + \mathrm{O}(x^{11})\right) \frac{d^4}{dx^4} y(x) + \mathrm{O}(x^7) \frac{d^3}{dx^3} y(x) + (x^2 + 2x^5 + \mathrm{O}(x^7)) \frac{d^2}{dx^2} y(x) + (3x + x^4 + \mathrm{O}(x^5)) \frac{d}{dx} y(x) + \mathrm{O}(x^6)\, y(x) = 0$$

$$\mathrm{O}(x^9) \left(\frac{d^5}{dx^5} y(x)\right) + \left(x^7 + \frac{81 x^9}{2} + 50 x^{10} + \mathrm{O}(x^{11})\right) \left(\frac{d^4}{dx^4} y(x)\right) + \mathrm{O}(x^7) \left(\frac{d^3}{dx^3} y(x)\right) + (2x^5 + x^2 + \mathrm{O}(x^7)) \left(\frac{d^2}{dx^2} y(x)\right) + (x^4 + 3x + \mathrm{O}(x^5)) \left(\frac{d}{dx} y(x)\right) + \mathrm{O}(x^6)\, y(x) = 0 \qquad (2)$$

An equation with both types of coefficients:

> $O(x^9)\,\frac{d^5}{dx^5}\,y(x)+\left(x^7+\sum_{k=9}^{\infty}\frac{k^2}{2}\,x^k\right)\frac{d^4}{dx^4}\,y(x)+\left(x^2+2\,x^5\right)\frac{d^2}{dx^2}\,y(x)+\left(3\,x+x^4+O(x^5)\right)\frac{d}{dx}\,y(x)+\left(\sum_{k=0}^{\infty}f(k)\,x^k\right)y(x)=0;$

$$O(x^9)\left(\frac{d^5}{dx^5}\,y(x)\right)+\left(x^7+\left(\sum_{k=9}^{\infty}\frac{k^2\,x^k}{2}\right)\right)\left(\frac{d^4}{dx^4}\,y(x)\right)+\left(2\,x^5+x^2\right)\left(\frac{d^2}{dx^2}\,y(x)\right)+\left(x^4+3\,x+O(x^5)\right)\left(\frac{d}{dx}\,y(x)\right)+\left(\sum_{k=0}^{\infty}f(k)\,x^k\right)y(x)=0 \quad (3)$$

>

LaurentSolution

Equations with completely given coefficients

For an ode whose coefficients are all algorithmically represented power series, the LaurentSolution command determines a finite set of all integers i_0 such that the ode has Laurent series solutions with valuation i_0, i.e. the ode has solutions in the form

$$\sum_{i=i_0}^{\infty}v(i)\,x^i \text{ where } v(i_0)\neq 0.$$

If the option 'top' = d is given (d is an integer), the LaurentSolution command for each valuation i_0 computes the initial terms of Laurent series solutions to the degree d or greater. The LaurentSolution command returns a list $[s_1, s_2, \ldots]$ of truncated Laurent series solutions for all found valuations. The elements of the list involve parameters of the form $\_c_1, \_c_2, \ldots$

For each element s_j these parameters can take any such values that the valuation of s_j does not change.

For the equation

> (1)

$$x^9\left(\frac{d^5}{dx^5}\,y(x)\right)+\left(x^7+\left(\sum_{k=9}^{\infty}\frac{k^2\,x^k}{2}\right)\right)\left(\frac{d^4}{dx^4}\,y(x)\right)+\left(2\,x^5+x^2\right)\left(\frac{d^2}{dx^2}\,y(x)\right)+\left(2\,x^{10}+x^4+3\,x\right)\left(\frac{d}{dx}\,y(x)\right)+\left(\sum_{k=0}^{\infty}f(k)\,x^k\right)y(x)=0 \quad (4)$$

we obtain

> $libname := \text{"maple.lib"}, libname:$
$with(TruncatedSeries):$
$LaurentSolution(\mathbf{(1)}, y(x), 'top'=3);$

$$\left[\frac{\_c_1}{x^2}+\_c_2-\frac{130\,x\,\_c_1}{3}+90\,x^2\,\_c_1-324\,x^3\,\_c_1+O(x^4),\ \_c_2+O(x^4)\right] \quad (5)$$

The truncation degree of the result may be d1 which is greater than d if it is needed to compute initial terms to the degree d1 to determine if Laurent solutions having valuation i_0 exist:

> $LaurentSolution(\mathbf{(1)}, y(x), 'top' = -1)$

$$\left[\frac{\_c_1}{x^2}+\_c_2+O(x),\ \_c_2+O(x)\right] \quad \textbf{(6)}$$

The same result will be if the option `'top' = d` is not given:

> *LaurentSolution*(**(1)**, $y(x)$);

$$\left[\frac{\_c_1}{x^2}+\_c_2+O(x),\ \_c_2+O(x)\right] \quad \textbf{(7)}$$

>

Equations with truncated coefficients

For an `ode` whose coefficients are all truncated series the `LaurentSolution` command investigates what can be learned from the `ode` about its solutions in the field of Laurent series. It constructs the maximal possible number of initial terms of solutions which are defined uniquely by known terms of coefficients of the `ode`. The greatest truncation degree of Laurent solutions is called the **threshold** of the `ode`.

For the equation

> **(2)**

$$O(x^9)\left(\frac{d^5}{dx^5}y(x)\right)+\left(x^7+\frac{81x^9}{2}+50x^{10}+O(x^{11})\right)\left(\frac{d^4}{dx^4}y(x)\right)+O(x^7)\left(\frac{d^3}{dx^3}y(x)\right)$$
$$+\left(2x^5+x^2+O(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(x^4+3x+O(x^5)\right)\left(\frac{d}{dx}y(x)\right)+O(x^6)\,y(x)=0 \quad \textbf{(8)}$$

we obtain the maximal possible number of initial terms for the solutions having valuation -2 (the first element of the returned list, the truncation degree is 1) and the maximal possible number of the initial terms for the solutions having valuation 0 (the second element of the returned list, the truncation degree is 5). The threshold for **(2)** is equal to 5:

> *LaurentSolution*(**(2)**, $y(x)$);

$$\left[\frac{\_c_1}{x^2}+\_c_2-\frac{130\,x\,\_c_1}{3}+O(x^2),\ \_c_2+O(x^6)\right] \quad \textbf{(9)}$$

If the option `'top' = d` is given, then the `LaurentSolution` command handles `d` as in the previous section.

> *LaurentSolution*(**(2)**, $y(x)$, '*top*' = 3, '*threshold*' = '*h*');

$$\left[\frac{\_c_1}{x^2}+\_c_2-\frac{130\,x\,\_c_1}{3}+O(x^2),\ \_c_2+O(x^4)\right] \quad \textbf{(10)}$$

Using the option `'threshold'='h'` (`h` is a name) we can obtain the information whether the given `d` is greater than the threshold of the `ode`. If it isn't then `h` is set equal to FAIL:

> *h*

FAIL **(11)**

Otherwise, `h` is set equal to the threshold:

> *LaurentSolution*(**(2)**, $y(x)$, '*top*' = 8, '*threshold*' = '*h*');
h;

$$\left[\frac{\_c_1}{x^2}+\_c_2-\frac{130\,x\,\_c_1}{3}+O(x^2),\ \_c_2+O(x^6)\right]$$

5 **(12)**

>

Equations with both types of coefficients

For an arbitrary `ode` whose coefficients are of both types, in general case it's impossible to determine the greatest degree of truncated Laurent solutions. The threshold of the `ode` may be a finite number or infinity. Then if the option `'top' = d` is not given, the `LaurentSolution` command computes exactly as many initial terms of the solutions as needed to find a set of all valuations of the Laurent solutions of the `ode`.
For the equation

> **(3)**

$$O(x^9)\left(\frac{d^5}{dx^5}y(x)\right)+\left(x^7+\left(\sum_{k=9}^{\infty}\frac{k^2x^k}{2}\right)\right)\left(\frac{d^4}{dx^4}y(x)\right)+(2x^5+x^2)\left(\frac{d^2}{dx^2}y(x)\right)+(x^4+3x+O(x^5))\left(\frac{d}{dx}y(x)\right)+\left(\sum_{k=0}^{\infty}f(k)x^k\right)y(x)=0 \quad \textbf{(13)}$$

we obtain

> $LaurentSolution(\textbf{(3)}, y(x));$

$$\left[\frac{\_c_1}{x^2}+\_c_2+O(x),\ \_c_2+O(x)\right] \quad \textbf{(14)}$$

If the option `'top' = d` is given, then the `LaurentSolution` command tries to compute all Laurent solutions to the truncation degree `d`. For **(3)** it's only possible for the solutions having valuation 0:

> $LaurentSolution(\textbf{(3)}, y(x), 'top' = 4);$

$$\left[\frac{\_c_1}{x^2}+\_c_2-\frac{130\,x\,\_c_1}{3}+O(x^2),\ \_c_2+O(x^5)\right] \quad \textbf{(15)}$$

If the threshold of the `ode` is greater than or equal to `d` and the option `'threshold'='h'` is given, then `h` is set equal to FAIL:

> $LaurentSolution(\textbf{(3)}, y(x), 'top' = 4, 'threshold' = 'h'):$
> $h;$

$$FAIL \quad \textbf{(16)}$$

In fact, the threshold of **(3)** is equal to ∞. This equation has the solution $y(x) = \_c_2$, where $\_c_2$ is an arbitrary constant. The trailing coefficient of **(3)** is $\sum_{k=0}^{\infty} f(k)\,x^k$ and the `LaurentSolution` command can check any finite number of values of $f(k)$:

> $\{seq(f(k), k = 0..100)\}$

$$\{0\} \quad \textbf{(17)}$$

but there is no algorithm to check that $f(k) = 0$ for all integer $k \geq 0$.

Equations which have no Laurent solution

If the given equation has no nonzero Laurent solution (the set of valuations of Laurent solutions is empty), then the `LaurentSolution` command returns the empty list:

> $LaurentSolution(x^2 diff(y(x), x) + y(x), y(x));$

$$[\ ] \quad \textbf{(18)}$$

and returns FAIL when the known terms of the coefficients of the given equation are not sufficient to find a set of valuations of Laurent solutions:

> $LaurentSolution(\mathrm{O}(x)\cdot diff(y(x),x)+y(x),y(x));$

$$FAIL \tag{19}$$

>

RegularSolution

For an ode with power series coefficients a formal regular solution is a finite sum of

$$x^{\lambda}\left(\sum_{k=0}^{m}\left(\sum_{i=i_{k,0}}^{\infty} v_k(i)\,x^i\right)\ln(x)^k\right)$$

where

- λ is an algebraic number,
- m is a non-negative integer,
- $i_{0,0},\ldots,i_{m,0}$ are integers and
- $v_k(i_{k,0})\neq 0$.

As for the case of Laurent solutions, the definition of the threshold of the equation is introduced. The RegularSolution command works similarly to the LaurentSolution command.

Examples

Below, we obtain the truncated regular solutions with $\lambda=0$ (the truncation degree is 4) and $\lambda=\frac{1}{3}$ (the truncation degree is 1). The threshold is computed, it is equal to 4:

> $RegularSolution\big((-3+x+\mathrm{O}(x^2))\,\theta(y(x),x,2)+(1+x+\mathrm{O}(x^2))\,\theta(y(x),x,1)+(x^4+\mathrm{O}(x^5))\,y(x),y(x),'threshold'='h'\big);$
$h;$

$$\left[\_c_1+\frac{x^4\_c_1}{44}+\mathrm{O}(x^5)+x^{1/3}\left(\_c_2+\frac{x\_c_2}{9}+\mathrm{O}(x^2)\right)\right]$$
$$4 \tag{20}$$

>

The truncated regular solutions with $\lambda=0$, $m=1$ (the first element of the returned list, the valuations are 0, the truncation degrees are 1 and 3); $m=0$ (the second element of the returned list, the valuation is 0, the truncation degree is 3); $m=1$ (the third element of the returned list, the coefficient of the logarithm of degree 0 has the truncation degree 1 and the valuation which is greater than 1, the coefficient of the logarithm of degree 1 has the truncation degree 3 and the valuation 0). The threshold is computed, it is equal to 3:

> $RegularSolution\big((4+x+\mathrm{O}(x^2))\,\theta(y(x),x,2)+(\mathrm{O}(x^2))\,\theta(y(x),x,1)+(x^3+\mathrm{O}(x^4))\,y(x),\ y(x),\ 'threshold'='h'\big);$
$h;$

$$\left[\_c_2+\mathrm{O}(x^2)+\ln(x)\left(\_c_1-\frac{x^3\_c_1}{36}+\mathrm{O}(x^4)\right),\_c_2-\frac{x^3\_c_2}{36}+\mathrm{O}(x^4),\mathrm{O}(x^2)+\ln(x)\left(\_c_1-\frac{x^3\_c_1}{36}+\mathrm{O}(x^4)\right)\right]$$
$$3 \tag{21}$$

>

If the equation has at least one completely given coefficient (below it is the coefficient of $\theta(y(x), x, 1)$ which is equal to 0) then we can use the command with different values of d in the option 'top' = d to obtain the threshold:

```
> eq := (4 + x + O(x^2)) θ(y(x), x, 2) + (x^3 + O(x^4)) y(x) :
  RegularSolution(eq, y(x), 'top' = 2, 'threshold' = 'h');
  h;
```

$$\left[\_c_2 + O(x^3) + \ln(x)\left(\_c_1 + O(x^3)\right), \_c_2 + O(x^3), O(x^3) + \ln(x)\left(\_c_1 + O(x^3)\right)\right]$$

FAIL **(22)**

```
> RegularSolution(eq, y(x), 'top' = 4, 'threshold' = 'h');
  h;
```

$$\left[\_c_2 + x^3\left(-\frac{\_c_2}{36} + \frac{\_c_1}{54}\right) + O(x^4) + \ln(x)\left(\_c_1 - \frac{x^3\,\_c_1}{36} + O(x^4)\right), \_c_2 - \frac{x^3\,\_c_2}{36} + O(x^4), \frac{x^3\,\_c_1}{54} + O(x^4) + \ln(x)\left(\_c_1 - \frac{x^3\,\_c_1}{36} + O(x^4)\right)\right]$$

3 **(23)**

The equation has no nonzero formal regular solution (the set of possible λ of regular solutions is empty):

```
> RegularSolution(x·θ(y(x), x, 2) + y(x), y(x))
```

[ ] **(24)**

The known terms of the coefficients are not sufficient to determine the set of λ:

```
> RegularSolution(x·θ(y(x), x, 2) + O(1)·θ(y(x), x, 1) + y(x), y(x))
```

FAIL **(25)**

>

FormalSolution

A formal exponential-logarithmic solution has a form

$$e^{Q(x)}\, x^{\lambda}\left(\sum_{k=0}^{m}\left(\sum_{i=i_{k,0}}^{\infty} v_k(i)\, x^{\frac{i}{q}}\right)\ln(x)^k\right)$$

- q is a positive integer,
- $Q(x)$ is a polynomial in $x^{-\frac{1}{q}}$,
- λ is an algebraic number,
- m is a non-negative integer,
- $i_{0,0}, \ldots, i_{m,0}$ are integers and $v_k(i_{k,0}) \neq 0$.

To construct all formal solutions for an ode with completely given coefficients the DEtools[formal_sol] or dsolve/series commands can be used. For an ode, whose coefficients may be truncated series the FormalSolution command computes the maximal possible terms of the exponent $Q(x)$. If $Q(x)$ is obtained completely, the FormalSolution command tries to compute initial terms of series which are components of solutions.

Example 1

For the next equation, the given initial terms are only sufficient to obtain the one term of the exponent $Q(x)$. The unknown part of solutions is denoted $y_1(x)$:

> $(x^3+O(x^4))\left(\frac{d}{dx}\,y(x)\right)+(2+O(x))\,y(x);$

$$(x^3+O(x^4))\left(\frac{d}{dx}\,y(x)\right)+(2+O(x))\,y(x) \qquad \textbf{(26)}$$

> $FormalSolution(\%, y(x));$

$$\left[e^{\frac{1}{x^2}}\,y_1(x)\right] \qquad \textbf{(27)}$$

The next equation is a **prolongation** of **(26)** (extra new known terms are added to the series coefficients). We obtain the second term of the exponent $Q(x)$. The notation $y_{reg}(x)$ in the result means that the exponential part of formal solutions is obtained completely:

> $(x^3+x^4+O(x^5))\left(\frac{d}{dx}\,y(x)\right)+(2+x+O(x^2))\,y(x):$
$FormalSolution(\%, y(x));$

$$\left[e^{\frac{1}{x^2}-\frac{1}{x}}\,y_{reg}(x)\right] \qquad \textbf{(28)}$$

Another prolongation of **(26)** leads to another result (both **(27)** and **(28)** are the prolongations of **(26)**; it shows that **(26)** presents the maximal possible information about the solution which is invariant to the prolongation of **(25)):**

> $\left(x^3+\frac{1}{2}\cdot x^4+O(x^5)\right)\left(\frac{d}{dx}\,y(x)\right)+(2+x+O(x^2))\,y(x):$
$FormalSolution(\%, y(x));$

$$\left[e^{\frac{1}{x^2}}\,y_{reg}(x)\right] \qquad \textbf{(29)}$$

Again and again, increasing the number of known terms in **(26)** we obtain more information about solutions:

> $(x^3+x^4+x^5+O(x^6))\left(\frac{d}{dx}\,y(x)\right)+(2+x-x^2+O(x^3))\,y(x):$
$FormalSolution(\%, y(x));$

$$\left[e^{\frac{1}{x^2}-\frac{1}{x}}\,x^2\,(\_c_1+O(x))\right] \qquad \textbf{(30)}$$

> $\left(x^3+x^4+x^5+\frac{3}{2}\cdot x^6+O(x^7)\right)\left(\frac{d}{dx}\,y(x)\right)+(2+x-x^2+O(x^4))\,y(x):$
$FormalSolution(\%, y(x));$

$$\left[e^{\frac{1}{x^2}-\frac{1}{x}}\,x^2\,(\_c_1+O(x^2))\right] \qquad \textbf{(31)}$$

> $\left(x^3+x^4+x^5+\frac{3}{2}\cdot x^6+\frac{1}{4}\cdot x^7+O(x^8)\right)\left(\frac{d}{dx}\,y(x)\right)+(2+x-x^2+O(x^5))\,y(x):$
$FormalSolution(\%, y(x));$

$$\left[e^{\frac{1}{x^2}-\frac{1}{x}}\,x^2\left(\_c_1-\frac{3\,\_c_1\,x^2}{2}+O(x^3)\right)\right] \qquad \textbf{(32)}$$

>

Example 2

The result of the FormalSolution command may contain following expressions: $y_{reg}\left(x^{\frac{1}{q}}\right)$, $y_{irr(p)}(x)$, $y[irr](x)$, $y_i(x)$, where q and i are positive integers, p is a rational number.

The notation $y_{reg}\left(x^{\frac{1}{q}}\right)$ in the result means that the exponent $Q(x)$ (together with the number q) is obtained completely but the algebraic number λ is not invariant to the all possible prolongation of the given equation (see **(28)** and **(29)** where $q=1$).
For example,

$$> \left(x^5+O(x^6)\right)\left(\frac{d^3}{dx^3}y(x)\right)+\left(-3x^3+O(x^4)\right)\left(\frac{d^2}{dx^2}y(x)\right)+O(x)\left(\frac{d}{dx}y(x)\right)+(2+O(x))\,y(x)=0:$$

$FormalSolution(\%, y(x));$

$$\left[y_1(x)+y_{irr}(x)+y_{irr(1)}(x)\right] \qquad \textbf{(33)}$$

The first term $y_1(x)$ of the result means that there are prolongations of the given equation that have a one-dimensional space of regular solutions, and there are prolongations that do not have regular solutions. The second term $y_{irr}(x)$ means that all prolongations of the given equation have such irregular solutions that the exponent $Q(x)$ has no invariant terms. The last term $y_{irr(1)}(x)$ means that all extensions of the given equation have at least a one-dimensional space of irregular solutions with an exponent $\frac{b}{x}+\ldots$, where $b\neq 0$ is not invariant algebraic number.
Here are two prolongations of the given equation confirming the above:

$$> \left(x^5+O(x^6)\right)\left(\frac{d^3}{dx^3}y(x)\right)+\left(-3x^3+O(x^4)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(2x+O(x^2)\right)\left(\frac{d}{dx}y(x)\right)+(1+O(x))\,y(x)=0:$$

$FormalSolution(\%, y(x));$

$$\left[\frac{\_c_1+O(x)}{\sqrt{x}}+e^{-\frac{2}{x}}y_{reg,1}(x)+e^{-\frac{1}{x}}y_{reg,2}(x)\right] \qquad \textbf{(34)}$$

$$> \left(x^5+O(x^6)\right)\left(\frac{d^3}{dx^3}y(x)\right)+\left(-3x^3+O(x^4)\right)\left(\frac{d^2}{dx^2}y(x)\right)+O(x^2)\left(\frac{d}{dx}y(x)\right)+(1+O(x))\,y(x)=0:$$

$FormalSolution(\%, y(x));$

$$\left[e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=1)}{\sqrt{x}}}y_{reg,1}(\sqrt{x})+e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=2)}{\sqrt{x}}}y_{reg,2}(\sqrt{x})+e^{-\frac{3}{x}}y_{reg,3}(x)\right] \qquad \textbf{(35)}$$

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References

Abramov, S., Khmelnov, D., Ryabenko, A.: Linear ordinary differential equations and truncated series. Comput. Math. Math. Phys. **49**(10), 1649–1659 (2019)

Abramov, S., Khmelnov, D., Ryabenko, A.: Regular solutions of linear ordinary differential equations and truncated series.
Comput. Math. Math. Phys. **60**(1), 2–15 (2020)

Abramov, S., Khmelnov, D., Ryabenko, A.: Procedures for searching Laurent and regular solutions of linear differential equations with the coefficients in the form of truncated power series.
Progr. and Comp. Soft. **46**(2), 67–75 (2020)

Abramov, S., Khmelnov, D., Ryabenko, A.: Truncated series and formal exponential-logarithmic solutions of linear ordinary differential equations.
Comput. Math. Math. Phys. **60**(10), 1609–1620 (2020)

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