Hypergeometric Solutions of First-Order Linear Difference Systems with Rational-Function Coefficients

S.A. Abramov, M. Petkovšek, A.A. Ryabenko

CC RAS, Ljubljana University

The problem

We consider a system of the form

$$\begin{cases} y_1(x+1) & = & a_{1,1}(x)y_1(x) + \cdots + a_{1,m}(x)y_m(x) \\ & \dots & \\ y_m(x+1) & = & a_{m,1}(x)y_1(x) + \cdots + a_{m,m}(x)y_m(x) \end{cases}$$

with $a_{i,j}(x) \in K(x)$, where K is an algebraically closed field of characteristic 0.

A short notation:

$$y(x+1) = A(x)y(x)$$

where $A(x)$ is an $(m \times m)$-matrix, and $y(x) = (y_1(x), \ldots, y_m(x))^T$.

$h(x)$ is called a hypergeometric term over K if $\dfrac{h(x+1)}{h(x)} \in K(x)$.

For example,

$$h(x) = (-1)^x : \quad \frac{h(x+1)}{h(x)} = -1.$$

$$h(x) = x! = \Gamma(x+1) : \quad \frac{h(x+1)}{h(x)} = x+1.$$

$$h(x) = \prod_{k=n_0}^{x-1} r(k) : \quad \frac{h(x+1)}{h(x)} = r(x).$$

Denote by H_K the K-linear space of finite linear combinations of hypergeometric terms over K with coefficients in K.

A basis of solutions belonging to H_K^m of

$$y(x+1) = A(x)y(x)$$

consists of elements of the form

$$h(x)R(x),$$

where $h(x)$ is a hypergeometric term and $R(x) \in K(x)^m$.

We propose an algorithm to find such basis.

The cyclic-vector method

Let $c^{[0]}(x)$ be a row vector with m random entries.

Compute

$$\begin{array}{lcl} c^{[1]}(x) & = & c^{[0]}(x+1)\,A(x); \\ c^{[2]}(x) & = & c^{[1]}(x+1)\,A(x); \\ & \dots & \\ c^{[m-1]}(x) & = & c^{[m-2]}(x+1)\,A(x). \end{array}$$

Set

$$B(x) = \begin{pmatrix} c^{[0]}(x) \\ c^{[1]}(x) \\ \vdots \\ c^{[m-1]}(x) \end{pmatrix}.$$

If $B(x)$ is an invertible matrix then $c^{[0]}(x)$ is *a cyclic vector* and

$$B(x+1)\,A(x)\,B^{-1}(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ u_0(x) & u_1(x) & u_2(x) & \dots & u_{m-1}(x) \end{pmatrix}$$

is *a companion matrix* for $A(x)$.

The system $y(x+1) = A(x)y(x)$ is equivalent to

$$v(x+m) = u_0(x)v(x) + \cdots + u_{m-1}(x)v(x+m-1).$$

For any $v(x) \in H_K(x)$ we get $y(x) \in H_K^m$ by

$$B(x)y(x) = \begin{pmatrix} v(x) \\ v(x+1) \\ \vdots \\ v(x+m-1) \end{pmatrix}.$$

– M. Petkovšek. Hypergeometric Solutions of Recurrences with Polynomial Coefficients. *Symbolic Computation*. 1992.

– M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. *J. Pure Appl. Algebra*. 1999.

For

$$B(x+1)\,A(x)\,B^{-1}(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ u_0(x) & u_1(x) & u_2(x) & \dots & u_{m-1}(x) \end{pmatrix}$$

we can compute $u(x) = (u_0(x), \dots, u_{m-1}(x))$ by

$$u(x)B(x) = c^{[m-1]}(x+1)A(x).$$

A. Bostan, F. Chyzak, E. de Panafieu. Complexity estimates for two uncoupling algorithms, *ISSAC'2013 Proceedings* , 85–92.

The resolving equation and matrix

Let $c^{[0]}(x) = (1, \underbrace{0, \ldots, 0}_{m-1})$.

Compute

$$\begin{array}{lcl} c^{[1]}(x) & = & c^{[0]}(x+1)\,A(x); \\ c^{[2]}(x) & = & c^{[1]}(x+1)\,A(x); \\ & \ldots & \\ c^{[k-1]}(x) & = & c^{[k-2]}(x+1)\,A(x); \\ c^{[k]}(x) & = & c^{[k-1]}(x+1)\,A(x). \end{array}$$

Set

$$B(x) = \begin{pmatrix} c^{[0]}(x) \\ c^{[1]}(x) \\ \vdots \\ c^{[k-1]}(x) \end{pmatrix}.$$

k is the least integer ($1 \le k \le m$) such that $c^{[0]}, c^{[1]}, \ldots, c^{[k]}$ are linearly dependent over $K(x)$:

$$u_0(x)c^{[0]}(x) + u_1(x)c^{[1]}(x) + \cdots + u_k(x)c^{[k]}(x) = 0.$$

The equation

$$u_0(x)y_1(x) + u_1(x)y_1(x+1) + \cdots + u_k(x)y_1(x+k) = 0$$

is called the y_1-*resolving equation*, and the $(k \times m)$-matrix $B(x)$ is called the y_1-*resolving matrix*.

The case of $k = m$

Find a basis of all hypergeometric solutions for y_1-resolving equation:

$$h_1(x), \ldots, h_{k_1}(x), \quad k_1 \leq m.$$

For each $h_j(x)$ we get

$$B(x)y(x) = \begin{pmatrix} h_j(x) \\ h_j(x+1) \\ \vdots \\ h_j(x+m-1) \end{pmatrix}.$$

If $\frac{h_j(x+1)}{h_j(x)} = r_j(x)$ then $y(x) = h_j(x)z(x)$ and

$$B(x)z(x) = \begin{pmatrix} 1 \\ r_j(x) \\ r_j(x)r_j(x+1) \\ \vdots \\ r_j(x)\cdots r_j(x+m-2) \end{pmatrix}.$$

The case of $k < m$

By y_1-resolving equation, we can find all solutions of

$$y(x+1) = A(x)y(x) \quad \text{such that } y_1(x) \neq 0.$$

For $y_1(x) = 0$, we have

$$B(x)y(x) = 0.$$

There exist $m-k$ entries $y_{i_1}(x), \dots, y_{i_{m-k}}(x)$ such that the other k entries can be expressed as linear forms in them.
The vector $\tilde{y}(x) = (y_{i_1}(x), \dots, y_{i_{m-k}}(x))^T$ satisfies

$$\tilde{y}(x+1) = \tilde{A}(x)\tilde{y}(x),$$

where $\tilde{A}(x)$ is an $(m-k) \times (m-k)$-matrix.

Compute a *resolving sequence* of equations and find their basises of solutions belonging to H_K: $h_1(x), \ldots, h_{k_1}(x)$.
Let

$$\tilde{h}_1(x), \ldots, \tilde{h}_{k_2}(x)$$

be all non-similar hypergeometric terms from $h_1(x), \ldots, h_{k_1}(x)$:

$$\frac{\tilde{h}_i(x)}{\tilde{h}_j(x)} \notin K(x), \quad i \neq j.$$

For each $\tilde{h}_j(x)$ substitute

$$y(x) = \tilde{h}_j(x) z(x)$$

into the given system, where $z(x) = (z_1(x), \ldots, z_m(x))^T \in K(x)^m$.

If $\frac{\tilde{h}_j(x+1)}{\tilde{h}_j(x)} = r_j(x)$ then we get a new system

$$z(x+1) = \frac{1}{r_j(x)} A(x)\, z(x).$$

– S. Abramov, M. Barkatou. Rational solutions of first order linear difference systems. ISSAC'98.

– S. Abramov, A. Gheffar, D. Khmelnov. Factorization of polynomials and gcd computations for finding universal denominators. CASC'2010.

– S. Abramov, A. Gheffar, D. Khmelnov. Rational Solutions of linear dfference equations: universal denominators and denominator bounds. Programming and Computer Software. 2011.

If $R_{j,1}(x), \ldots, R_{j,s_j}(x) \in K(x)^m$ is a basis for rational solutions of

$$z(x+1) = \frac{1}{r_j(x)} A(x)\, z(x)$$

then we obtain K-linearly independent hypergeometric solutions

$$\tilde{h}_j(x) R_{j,1}(x), \ldots, \tilde{h}_j(x) R_{j,s_j}(x)$$

of $y(x+1) = A(x)\, y(x)$. Consider all such $\tilde{h}_j(x)$ for $j = 1, \ldots, k_2$. The solutions

$$\tilde{h}_1(x) R_{1,1}(x), \ldots, \tilde{h}_1(x) R_{1,s_1}(x), \;\; \ldots, \\ \tilde{h}_{k_2}(x) R_{k_2,1}(x), \ldots, \tilde{h}_{k_2}(x) R_{k_2,s_{k_2}}(x)$$

generate over K all the solutions of the given system which have the form $h(x)R(x)$.

The algorithm is implemented in Maple 18 as the procedure **HypergeometricSolution** in a package **LRS** (Linear Recurrence Systems) (available on *http://www.ccas.ru/ca/doku.php/lrs*).

Besides the resolving procedure, we implemented also the search for hypergeometric solutions based on the cyclic vector approach. We tested **HypergeometricSolution** for various systems and compared the CPU time for the resolving and the cyclic-vector methods.

For systems with $A(x)$ of size 4×4 and 16×16 the CPU time is

0.303 and 345.046 sec

(our algorithm) vs

0.410 and 1063.747 sec

(the cyclic vector).