On a Simple Lower Bound for the Matrix Rank

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The 5th International Conference
"Computer Algebra"
June 26–28, 2023

Simple lower bounds are important for planning calculations because a sufficiently large rank ensures the applicability of some algorithms. The rank of an $n \times n$ matrix over a field can be calculated using a polynomial number of processors and performing only $O(\log_2^2 n)$ algebraic operations per processor (Chistov, 1985). The matrix rank is as hard as matrix multiplication (Cheung, Kwok, and Lau, 2013).

In practice, calculating the matrix rank requires a lot of time. Unfortunately, the rank calculation method implemented using the NumPy library, which is based on the singular value decomposition, is not suitable because of possible errors associated with the use of floating point numbers.

Chistov A.L. Fast parallel calculation of the rank of matrices over a field of arbitrary characteristic. In: L. Budach (eds) Fundamentals of Computation Theory. FCT 1985. Lecture Notes in Computer Science, vol. 199. Springer, Heidelberg, 1985, pp. 63–69.

Cheung H.Y., Kwok T.C., Lau L.C. Fast matrix rank algorithms and applications. J. ACM. 2013. Vol. 60, N. 5, Article 31, pp. 1–25.

To show the computational complexity of the matrix rank over $\mathbb{Z}$, we have used Python 3.10.4 and NumPy 1.22.4. The calculations were carried out on a personal computer with Intel® Core i5-3570 and 16 GB RAM.

Time (in seconds) it takes to calculate the rank of a random $n \times n$ matrix.

Order n	105	120	136	153	171	190	210	231
Time (s)	6	11	18	30	50	79	120	190

The listings with examples are available at `http://lab6.iitp.ru/-/qualg`

Zverkov O.A. *et al.* Effective lower bounds on the matrix rank and their applications. Programming and Computer Software. 2023. (Submitted).

Let us denote by K an arbitrary field of characteristic not equal to two. Let us consider an $n \times n$ matrix over the field K, where every entry outside the leading diagonal belongs to the set $\{0, 1\}$, but every diagonal entry is neither 0 nor 1. **How small can its rank be?**

This problem has a simple geometric interpretation. Let us consider an affine space over the field K with a fixed system of Cartesian coordinates. A point is identified with a column, where entries are coordinates of the point in this coordinate system. A column of zeros and ones corresponds to a $(0, 1)$-point, i.e., to a vertex of the unit cube. In matrices under consideration, each column corresponds to a point in a straight line passing through two adjacent $(0, 1)$-points, but this point does not coincide with any $(0, 1)$-point. Moreover, different columns of the matrix correspond to non-parallel straight lines.

Example. For the 3×3 matrix having neither 0 nor 1 in the leading diagonal

$$\begin{pmatrix} 1/2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

the rank equals two. Three columns correspond to three points belonging to a straight line L. The straight line L is given by the system of two equations $1 - 2x_1 + x_2 = 0$ and $x_2 + x_3 = 0$. But the straight line L does not pass through any $(0,1)$-point.

The rank of a matrix A is related to the dimensionality of the affine hull L of all points corresponding to columns of A. If L passes through the origin, then $\text{rank}(A) = \dim(L)$, else $\text{rank}(A) = \dim(L) + 1$.

Theorem 1. *Given an $n \times n$ matrix A over the field K, where every entry outside the leading diagonal belongs to the set $\{0,1\}$, but every diagonal entry is neither 0 nor 1. The rank of the matrix A is at least $n/2$.*

The lower bound is tight. Let $\lceil \cdot \rceil$ denote rounding up.

Theorem 2. *For every odd n, there is an $n \times n$ matrix A over the field K such that every entry outside the leading diagonal belongs to the set $\{0, 1\}$, every diagonal entry is neither 0 nor 1, no $(0,1)$-point belongs to the affine hull of all points corresponding to columns of the matrix A, and the equality $\operatorname{rank}(A) = \lceil n/2 \rceil$ holds.*

Proof. Let us consider the $n \times n$ matrix

$$A = \left(\begin{array}{c|ccccccc} 1/2 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ \hline 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{array}\right).$$

Let us denote by B a $(n-1) \times (n-1)$ matrix obtained by removing both first column and first row from the matrix A.
Obviously, $\operatorname{rank}(A) = \operatorname{rank}(B) + 1$. The matrix B is block-diagonal with 2×2 blocks. All blocks are degenerate. Thus, $\operatorname{rank}(B) = (n-1)/2$.
Next, $\operatorname{rank}(A) = \operatorname{rank}(B) + 1 = (n+1)/2 = \lceil n/2 \rceil$.

Every column of the matrix

$$A = \begin{pmatrix} 1/2 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

is a solution to the inhomogeneous system of equations

$$\left\{ \begin{array}{rcl} 2x_1 - x_2 - \cdots - x_{2k} - \cdots - x_{n-1} & = & 1 \\ x_{2k} + x_{2k+1} & = & 0, \;\; 1 \leq k \leq (n-1)/2 \end{array} \right.$$

This system defines the affine hull, which does not pass through any $(0,1)$-point. $\square$

Theorem 3. *Given an even n and an $n \times n$ matrix A over the field K, where every entry outside the leading diagonal belongs to the set $\{0, 1\}$, but every diagonal entry is neither 0 nor 1. If no $(0, 1)$-point belongs to the affine hull of all points corresponding to columns of the matrix A, then the rank of the matrix A is at least $(n/2) + 1$.*

Example. For 2×2 matrices under consideration, the rank equals one for matrices having reciprocal entries in the leading diagonal

$$\begin{pmatrix} 1/\alpha & 1 \\ 1 & \alpha \end{pmatrix},$$

where $\alpha \notin \{0, 1\}$. Two points corresponding to columns of this matrix belong to a straight line that passes through the origin, i.e., through a $(0, 1)$-point. This straight line is given by the equation $x_2 = \alpha x_1$.
So, if no $(0, 1)$-point belongs to the affine hull of all points corresponding to columns of the matrix A, then $\text{rank}(A) = 2$.

For $n > m$, a projection $K^n \to K^m$ is so-called **orthographic** when the projection forgets some coordinates.
It is important that the image of any $(0, 1)$-point is again a $(0, 1)$-point. The term was historically used to denote orthogonal projections from three-dimensional space onto a plane over reals.

Let $\lfloor \cdot \rfloor$ denote rounding down.

Corollary 1. *For subspaces $L \subset K^n$, if $\dim L < \lfloor n/2 \rfloor$ and L does not pass through any $(0, 1)$-point, then there is an orthographic projection onto some coordinate hyperplane such that the image of L does not pass through any $(0, 1)$-point.*

Corollary 2. *Given a positive integer s. There is an s-dimensional affine subspace $L \subset K^{2s+1}$ such that L does not pass through any $(0, 1)$-point, but under the orthographic projection onto any coordinate hyperplane, the image of L passes through some $(0, 1)$-point.*

Thank you!

The author is also grateful to Alexey A. Boykov
(MIREA — Russian Technological University)
for useful comments on descriptive geometry.