

```

> restart;
>
y(x+1)=A.y(x)
> A := 
$$\begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)x} & \frac{x^2+x+1}{x} \end{bmatrix};$$


http://www.ccas.ru/ca/resolvingsequence
> read "resolvingsequence.mpl";
> RS:-ResolvingSequence(A, y(x), OreTools:-SetOreRing(x,'shift')));

$$\left[ \begin{aligned} &(-x^6 - x^5 + 4x^4 + 3x^3 - 3x^2 - 2x) y_1(x) + (2x^6 + 6x^5 + 2x^4 - 6x^3 - 6x^2 - 4x) y_1(x) \\ &+ 1) + (-x^6 - 6x^5 - 11x^4 - 2x^3 + 8x^2 + 8x + 4) y_1(x+2) + (x^5 + 4x^4 + x^3 \\ &- 6x^2) y_1(x+3) = 0, -x y_4(x) + y_4(x+1) = 0 \end{aligned} \right] \quad (1)$$


> LREtools:-hypergeomsols(-x y_4(x) + y_4(x+1) = 0, y[4](x), { }, output = basis)

$$[\Gamma(x)] \quad (2)$$


http://www.ccas.ru/ca/lrs
> read "lrshypergeomsols.mpl";
> LRS:-HypergeometricSolution(A, x);

$$\begin{bmatrix} 0 \\ -\Gamma(x) \\ 0 \\ \Gamma(x) \end{bmatrix} \quad (3)$$


>
> ResolvingMatrix( ) [1];

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2(x^2+x-1)}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} \quad (4)$$


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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2(x^2+x-1)}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} :$$

> *ResolvingDependence*() [1];

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(5)

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} ;$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -y_4(x) \\ 0 \\ y_4(x) \end{bmatrix}$$

(6)

> *L* := *ResolvingEquation*(*A*, *y*(*x*), *OreTools*:-*SetOreRing*(*x*, 'shift'));

$$\begin{aligned} L := & (-x^6 - x^5 + 4x^4 + 3x^3 - 3x^2 - 2x) y_1(x) + (2x^6 + 6x^5 + 2x^4 - 6x^3 - 6x^2 \\ & - 4x) y_1(x+1) + (-x^6 - 6x^5 - 11x^4 - 2x^3 + 8x^2 + 8x + 4) y_1(x+2) + (x^5 + 4x^4 \\ & + x^3 - 6x^2) y_1(x+3) = 0 \end{aligned}$$

(7)

> *m* := 4 : *i* := *RS*:-*Indicator*() [1] :

Y := *Vector*([*seq*(*y*[*k*](*x*), *k* = 1 .. *m*)]) ;

B := *ResolvingMatrix*() [1] :

Rhs := *Vector*([*seq*(*eval*(*Y*[*i*], *x* = *x* + *k*), *k* = 0 .. *LinearAlgebra*:-*RowDimension*(*B*) - 1)]) :
Y[*i*] ≠ 0;

L, *Sol* := *LREtools*:-*hypergeomsols*(*L*, *Y*[*i*], { }, *output* = *basis*);

sbst := { *seq*(*eval*(*Y*[*i*] = *Sol*[1], *x* = *x* + *k*), *k* = 0 .. *LinearAlgebra*:-*RowDimension*(*B*) - 1) } :

'(' *B*, *Y*) = *Rhs*;

'(' *B*, *eval*(*Y*, *sbst*)) = *eval*(*Rhs*, *sbst*);

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Y := Vector( [y[2](x), y[3](x), y[4](x)] ) :
'!(B[1..-1, 2..-1], Y) = map(radnormal, eval(Rhs - B[1..-1, 1..1]. Vector( [y[1](x)] ),
    sbst));
slv := collect(solve( convert( (lhs - rhs) (%), set), convert(Y, set) ), convert(Y, set) ) :
"-----";
slv[1]; slv[2]; slv[3];
"-----";
y(x + 1) = A.y(x);
eval(Y, x = x + 1) = '(A[2..-1, 2..-1], Y) + LinearAlgebra:-Column(A, i)[2..-1].y[1](x);
lhs(%) = map(collect, eval(rhs(%), sbst union slv), y[4](x), radnormal);
Sol2 := eval(LREtools:-dAlembertiansols( convert( (lhs - rhs) (%), list)[3], y[4](x), { },
    output = basis[C]), C[1] = 0);
slv := { y[1](x) = Sol[1], map(el → lhs(el) = eval(rhs(el), y[4](x) = Sol2), slv)[ ] } :
⟨y[1](x), Y⟩ :
% = eval(%, slv);
seq(value(eval( eval(rhs(%), x = x + 1) - A.rhs(%), x = xi)), xi = 2..10) :

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$$Y := \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

$$y_1(x) \neq 0$$

$$\begin{aligned} & (-x^6 - x^5 + 4x^4 + 3x^3 - 3x^2 - 2x) y_1(x) + (2x^6 + 6x^5 + 2x^4 - 6x^3 - 6x^2 - 4x) y_1(x \\ & + 1) + (-x^6 - 6x^5 - 11x^4 - 2x^3 + 8x^2 + 8x + 4) y_1(x + 2) + (x^5 + 4x^4 + x^3 \\ & - 6x^2) y_1(x + 3) = 0 \end{aligned}$$

$$Sol := \left[\frac{1}{x-1} \right]$$

$$\begin{aligned} & \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2(x^2+x-1)}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{array} \right] \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} \\ & = \begin{bmatrix} y_1(x) \\ y_1(x+1) \\ y_1(x+2) \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2(x^2+x-1)}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{x-1} \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x-1} \\ \frac{1}{x} \\ \frac{1}{x+1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{x-1}{x+1} & 0 \\ -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} \cdot \begin{bmatrix} y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{x}{(x+2)(x-1)} \end{bmatrix}$$

"-----"

$$y_2(x) = -y_4(x) + \frac{1}{x-1}$$

$$y_3(x) = 0$$

$$y_4(x) = y_4(x)$$

"-----"

$$y(x+1) = \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)x} & \frac{x^2+x+1}{x} \end{bmatrix} \cdot y(x)$$

$$\begin{bmatrix} y_2(x+1) \\ y_3(x+1) \\ y_4(x+1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{x+1} & -x \\ 1 & x-1 & 1 \\ \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)x} & \frac{x^2+x+1}{x} \end{bmatrix} \cdot \begin{bmatrix} y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} + \begin{bmatrix} y_1(x) \\ -y_1(x) \\ -\frac{y_1(x)(x+2)}{x} \end{bmatrix}$$

$$\begin{bmatrix} y_2(x+1) \\ y_3(x+1) \\ y_4(x+1) \end{bmatrix} = \begin{bmatrix} -x y_4(x) + \frac{1}{x-1} \\ 0 \\ x y_4(x) - \frac{1}{(x-1)x} \end{bmatrix}$$

$$Sol2 := -\Gamma(x) \left(\sum_{xI=2}^{x-1} \frac{1}{xI (xI-1) \Gamma(xI+1)} \right)$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{x-1} \\ \Gamma(x) \left(\sum_{xI=2}^{x-1} \frac{1}{xI (xI-1) \Gamma(xI+1)} \right) + \frac{1}{x-1} \\ 0 \\ -\Gamma(x) \left(\sum_{xI=2}^{x-1} \frac{1}{xI (xI-1) \Gamma(xI+1)} \right) \end{bmatrix}$$

(8)

> RS:-ResolvingSequence(A, y(x), OreTools:-SetOreRing(x,'shift')) :

y[1](x) = 0; Y := Vector([seq(y[k](x), k = 1 .. m)]) :

!(B, Y) = 0;

Ds := ResolvingDependence() [1] : y(x) = Ds[1].y(x) :

Y = !(Ds, Vector([y[1](x), y[2](x), y[3](x), y[4](x)])) ;

value(%);

Ss := ResolvingSystems() :

"-----";

y(x+1) = Ss[1].y(x); y[4](x+1) = Ss[2][1, 1].y[4](x)

y₁(x) = 0

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2(x^2+x-1)}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = 0$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

$$\begin{aligned}
& \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -y_4(x) \\ 0 \\ y_4(x) \end{bmatrix} \\
& \text{"-----"} \\
& y(x+1) = \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)x} & \frac{x^2+x+1}{x} \end{bmatrix} \cdot y(x) \\
& y_4(x+1) = x y_4(x)
\end{aligned} \tag{9}$$

$$\begin{aligned}
& \text{RS:-CyclicVector}(A, y(x), \text{OreTools:-SetOreRing}(x, \text{'shift'})); \\
& (216x^{16} + 2736x^{15} + 13848x^{14} + 13896x^{13} - 225046x^{12} - 1279404x^{11} - 2590626x^{10} \\
& + 148424x^9 + 9058465x^8 + 12241281x^7 - 182690x^6 - 11062671x^5 - 6480547x^4 \\
& - 119042x^3 + 461160x^2) u(x) + (-432x^{16} - 6552x^{15} - 41520x^{14} - 93696x^{13} \\
& + 464852x^{12} + 4572454x^{11} + 17338588x^{10} + 36072472x^9 + 38235226x^8 + 3556419x^7 \\
& - 41619459x^6 - 50072690x^5 - 27167229x^4 - 9544887x^3 - 3290586x^2 - 673640x) \\
& u(x+1) + (216x^{16} + 4248x^{15} + 34440x^{14} + 122616x^{13} - 152038x^{12} - 3935334x^{11} \\
& - 21138966x^{10} - 63858516x^9 - 116526301x^8 - 109403774x^7 + 17271111x^6 \\
& + 187694014x^5 + 233803892x^4 + 126789448x^3 + 13309796x^2 - 16616792x \\
& - 5806240) u(x+2) + (-432x^{15} - 6768x^{14} - 43104x^{13} - 94128x^{12} + 557732x^{11} \\
& + 5759796x^{10} + 24865732x^9 + 62152912x^8 + 85434160x^7 + 28885218x^6 \\
& - 104044129x^5 - 181250808x^4 - 117491893x^3 - 15097092x^2 + 17694964x \\
& + 6847280) u(x+3) + (216x^{14} + 2736x^{13} + 13128x^{12} + 8544x^{11} - 291958x^{10} \\
& - 2057116x^9 - 7030722x^8 - 12767046x^7 - 8438475x^6 + 11305591x^5 + 27629889x^4 \\
& + 20515827x^3 + 3232762x^2 - 3139056x - 1314720) u(x+4) = 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
& \text{RS:-ResolvingMatrix}() [1]; \\
& \left[\left[\left[4, 2, -5, -9 \right], \right. \right. \\
& \left. \left[\frac{2(10x+7)}{x}, -\frac{14x+9}{x}, -\frac{5x^3+13x^2-22x-9}{(x+1)x}, -\frac{11x^2+14x+9}{x} \right] \right],
\end{aligned} \tag{11}$$

$$\begin{aligned}
& \left[\frac{16x^4 + 114x^3 + 244x^2 + 215x + 68}{(x+1)(x+2)x}, -\frac{16x^4 + 97x^3 + 183x^2 + 161x + 68}{(x+1)(x+2)x}, \right. \\
& -\frac{5x^6 + 39x^5 + 81x^4 + 78x^3 - 19x^2 - 137x - 68}{(x+1)^2(x+2)x}, \\
& \left. -\frac{11x^5 + 60x^4 + 152x^3 + 205x^2 + 161x + 68}{(x+1)(x+2)x} \right], \\
& \left[\frac{16x^7 + 228x^6 + 1358x^5 + 4427x^4 + 8433x^3 + 9190x^2 + 5284x + 1314}{(x+2)^2(x+3)(x+1)x}, \right. \\
& -\frac{16x^7 + 217x^6 + 1220x^5 + 3759x^4 + 6885x^3 + 7493x^2 + 4594x + 1314}{(x+2)^2(x+3)(x+1)x}, \\
& -\frac{1}{(x+2)^2(x+3)(x+1)^2x} (5x^9 + 80x^8 + 488x^7 + 1633x^6 + 3295x^5 + 3652x^4 \\
& + 841x^3 - 3162x^2 - 3808x - 1314), \\
& \left. -\frac{11x^8 + 137x^7 + 756x^6 + 2463x^5 + 5321x^4 + 7897x^3 + 7757x^2 + 4594x + 1314}{(x+2)^2(x+3)(x+1)x} \right] \Bigg]
\end{aligned}$$



► $y(x+1) = A.y(x)$

