```
> restart;
>
> DeltaRing := OreTools:-SetOreRing(x, 'Delta',
    'sigma' = ((p, x) -> eval(p, x = x + 1)),
    'sigma_inverse' = ((p, x) -> eval(p, x = x - 1)),
    'delta' = ((p, x) -> eval(p, x = x + 1) - p),
   'theta1' = 0) :
```

$$> \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} . \Delta^{(2)}(y(x)) + \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix} . \Delta(y(x)) + \begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix} . y(x) = 0 :$$

http://www.ccas.ru/ca/resolvingsequence

```
> read "resolvingsequence.mpl" :
```

$$> RS\text{:-}ResolvingSequence\left(OrePoly\left(\begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix}, \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} \right), DeltaRing \right);$$

$Indicator(\,);$

$$[OrePoly(-x+1, 2x+3, -3, -2x-7, x+6), OrePoly(-1, 0, 1)]$$

$$[1, 3] \tag{1}$$

$$> (x+6)\,\Delta^{(4)}(y_1(x)) + (-2x-7)\,\Delta^{(3)}(y_1(x)) - 3\,\Delta^{(2)}(y_1(x)) + (2x+3)\,\Delta(y_1(x)) + (-x+1)\,y_1(x) = 0 :$$

$$> \Delta^{(2)}(y_3(x)) - y_3(x) = 0 :$$

```
>
>
>
> ResolvingDependence()[1];
```

(2)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

> $\begin{bmatrix} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix};$

$$\begin{bmatrix} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix} = \begin{bmatrix} 0 \\ y3(x) \\ 0 \\ \Delta(y3(x)) \end{bmatrix} \quad (3)$$

> $OreTools{:}\text{-}Apply(\ OrePoly(-1, 0, 1), y(x), DeltaRing) = 0;$
$Sols := LREtools[hypergeomsols](\%, y(x), \{\ \}, output = basis)$

$$y(x+2) - 2\,y(x+1) = 0$$

$$Sols := [2^x] \quad (4)$$

>

>

> $simplify\left(\left(\begin{bmatrix} -x & 2 & x+2 \\ 2\,x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 2^x \end{bmatrix} + \begin{bmatrix} x+1 & x+1 & 0 \\ -2\,x-4 & -2\,x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix} .map2\left(OreTools{:}\text{-}Apply, OrePoly(0, 1), \begin{bmatrix} 0 \\ 0 \\ 2^x \end{bmatrix}, DeltaRing\right) + \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2\,x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} .map2\left(OreTools{:}\text{-}Apply, OrePoly(0, 0, 1), \begin{bmatrix} 0 \\ 0 \\ 2^x \end{bmatrix}, DeltaRing\right)\right)\right) = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \quad (5)$$

>

> $DeltaRing := OreTools{:}\text{-}SetOreRing(x, \text{'Delta'},$
$\text{'sigma'} = ((p, x) \rightarrow eval(p, x = x+1)),$
$\text{'}sigma\_inverse\text{'} = ((p, x) \rightarrow eval(p, x = x-1)),$

```
   'delta' = (proc(p, x);
        if p :: ratpoly(anything, x) then
           eval(p, x = x + 1) - p
        elif op(0, p) = Delta then
           (Delta@@2)(op(p))
        elif op([0, 0], p) = '`@@`' and op([0, 1], p) = Delta then
           (Delta@@(op([0, 2], p) + 1))(op(p))
        else
           Delta(p)
        end if
   end proc),
   'theta1' = 0) :
```

```
> syst := OrePoly(Matrix([[-x, 2, x+2], [2 x, -2, -x-2], [x, 0, -x-2]]),
                  Matrix([[x+1, x+1, 0], [-2 x-4, -2 x-4, 0], [-x-3, -x-3, 0]]),
                  Matrix([[1, -x-5, -x-2], [0, 2 x+10, x+2], [1, x+5, x+2]])) :
```

$$syst := OrePoly\left(\begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix}, \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix}\right):$$

```
> L := ResolvingSequence(syst, DeltaRing);
  Indicator( );
```

$$L := [OrePoly(-x+1, 2x+3, -3, -2x-7, x+6), OrePoly(-1, 0, 1)]$$

$$[1, 3] \tag{6}$$

```
> ResolvingMatrix( );
  m := 3 : i := Indicator( )[1] : n := 2 :
  B := ResolvingMatrix( )[1] :
  Y := Vector([y1(x), y2(x), y3(x), Delta(y1(x)), Delta(y2(x)), Delta(y3(x))]) :
  Y[i] ≠ 0;
  OreTools:-Apply( L[1], Y[i], DeltaRing) = 0;
  Rhs := Matrix(LinearAlgebra:-RowDimension(B), 1, [Y[i], seq((Delta@@k)(Y[i]), k = 1
       ..LinearAlgebra:-RowDimension(B) - 1)]) :
  '.'(B, Y) = Rhs;
  '.'(LinearAlgebra:-DeleteColumn(B[n + 1 ..-1], [seq(i + m*k, k = 0 ..m·(n - 1))]),
  LinearAlgebra:-DeleteRow(Y, [seq(i + m*k, k = 0 ..n-1)])) =
  - '.'(⟨LinearAlgebra:-SubMatrix(B[n + 1 ..-1], [1 ..-1], [seq(i + m*k, k = 0 ..n - 1)])⟩,
  LinearAlgebra:-SubMatrix(Y, [seq(i + m*k, k = 0 ..n - 1)], [1 ..-1])) + Rhs[n + 1 ..-1];
  value(%);
  "----------------------";
 slv := collect( solve(convert((lhs - rhs)(%%), set), {y2(x), y3(x), Δ(y2(x))}), {y1(x),
      y2(x), y3(x), Δ(y1(x)), Δ(y2(x)), Δ(y3(x))}) :
  slv[1];
  slv[2];
  slv[3];
  "----------------------";
  '.'(op(3, syst), Vector([(Delta@@2)(y1(x)), (Delta@@2)(y2(x)), (Delta@@3)(y3(x))]))
      + '.'(op(2, syst), Y[m + 1 ..2·m]) + + '.'(op(1, syst), Y[1 ..m]) = 0;
```

$$\left[\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ -\frac{x}{x+5} & -\frac{x+4}{x+5} & 0 & \frac{2x+7}{x+5} & \frac{x+2}{x+5} & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right]$$

$$y1(x) \neq 0$$

$$(-x+1)\, y1(x) + (2x+3)\, \Delta(y1(x)) - 3\, \Delta^{(2)}(y1(x)) + (-2x-7)\, \Delta^{(3)}(y1(x)) + (x+6)\, \Delta^{(4)}(y1(x)) = 0$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ -\frac{x}{x+5} & -\frac{x+4}{x+5} & 0 & \frac{2x+7}{x+5} & \frac{x+2}{x+5} & 0 \end{array}\right] \cdot \left[\begin{array}{c} y1(x) \\ y2(x) \\ y3(x) \\ \Delta(y1(x)) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{array}\right] = \left[\begin{array}{c} y1(x) \\ \Delta(y1(x)) \\ \Delta^{(2)}(y1(x)) \\ \Delta^{(3)}(y1(x)) \end{array}\right]$$

$$\left[\begin{array}{cccc} -1 & 0 & 1 & 0 \\ -\frac{x+4}{x+5} & 0 & \frac{x+2}{x+5} & 0 \end{array}\right] \cdot \left[\begin{array}{c} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{array}\right] = -\left[\begin{array}{cc} 0 & 1 \\ -\frac{x}{x+5} & \frac{2x+7}{x+5} \end{array}\right] \cdot \left[\begin{array}{c} y1(x) \\ \Delta(y1(x)) \end{array}\right] + \left[\begin{array}{c} \Delta^{(2)}(y1(x)) \\ \Delta^{(3)}(y1(x)) \end{array}\right]$$

$$\left[\begin{array}{c} -y2(x) + \Delta(y2(x)) \\ -\frac{(x+4)\, y2(x)}{x+5} + \frac{(x+2)\, \Delta(y2(x))}{x+5} \end{array}\right] = \left[\begin{array}{c} -\Delta(y1(x)) + \Delta^{(2)}(y1(x)) \\ \frac{x\, y1(x)}{x+5} - \frac{(2x+7)\, \Delta(y1(x))}{x+5} + \Delta^{(3)}(y1(x)) \end{array}\right]$$

"-----------------------"

$$\Delta(y2(x)) = \left(\frac{1}{2}x + \frac{3}{2}\right) \Delta(y1(x)) + \frac{1}{2}x\, \Delta^{(2)}(y1(x)) - \frac{1}{2}x\, y1(x) - \frac{1}{2}\Delta^{(3)}(y1(x))\, x + 2\, \Delta^{(2)}(y1(x)) - \frac{5}{2}\Delta^{(3)}(y1(x))$$

$$y2(x) = \left(\frac{1}{2}x + \frac{5}{2}\right) \Delta(y1(x)) + \frac{1}{2}x\, \Delta^{(2)}(y1(x)) - \frac{1}{2}x\, y1(x) - \frac{1}{2}\Delta^{(3)}(y1(x))\, x$$

$$+\Delta^{(2)}(y1(x)) - \frac{5}{2}\Delta^{(3)}(y1(x))$$

$$y3(x) = y3(x)$$

"-----------------------"

$$\begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} \cdot \begin{bmatrix} \Delta^{(2)}(y1(x)) \\ \Delta^{(2)}(y2(x)) \\ \Delta^{(3)}(y3(x)) \end{bmatrix} + \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta(y1(x)) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix} \tag{7}$$

$$+ \begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix} \cdot \begin{bmatrix} y1(x) \\ y2(x) \\ y3(x) \end{bmatrix} = 0$$

```
> Y[i]=0;
  Ds := ResolvingDependence( );
  Y := Vector( [y2(x), y3(x), Delta(y2(x)), Delta(y3(x))]) :
  Y='.'(Ds[1], Y);
  value(%);
  Systs := ResolvingSystems( );
  "---------------------";
  Vector( [Delta(y3(x)), (Delta@@2)(y3(x))]) ='.'( Systs[2], Vector( [y3(x), Delta(y3(x))]));
  value(%);
```

$$y1(x) = 0$$

$$Ds := \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix}$$

$$\begin{bmatrix} y2(x) \\ y3(x) \\ \Delta(y2(x)) \\ \Delta(y3(x)) \end{bmatrix} = \begin{bmatrix} 0 \\ y3(x) \\ 0 \\ \Delta(y3(x)) \end{bmatrix}$$

$$Systs := \left[OrePoly\left(\begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix}, \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} \right), \right.$$

$$\left.\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$$

"---------------------"

$$\begin{bmatrix} \Delta(y3(x)) \\ \Delta^{(2)}(y3(x)) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y3(x) \\ \Delta(y3(x)) \end{bmatrix}$$

$$\begin{bmatrix} \Delta(y3(x)) \\ \Delta^{(2)}(y3(x)) \end{bmatrix} = \begin{bmatrix} \Delta(y3(x)) \\ y3(x) \end{bmatrix} \tag{8}$$

>