LiouvillianSolution - find Liouvillian solutions of Linear (q-)Recurrence equations

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Date created: March, 2008

Date created: for q-case December, 2012

Calling Sequence

LiouvillianSolution(***reqn***, ***fcn***, **'indices'**=[*n*,*k*], **'low'** = ***lb***)

Parameters

`reqn`	:	linear (q-)recurrence.
`fcn`	:	function name, ***v(n)*** for instance.
`'indices'=[n,k]`	:	optional; where ***n***, ***k*** are names.
`'low'=lb`	:	optional; where ***lb*** is a name

Returns

- the Liouvillian solution of the given linear (q-)recurrence ***reqn*** or 0 if there is no such solution.

Description

- The procedure finds the Liouvillian solution of the given linear (q-)recurrence ***reqn*** in ***fcn*** with the rational function coefficients using the algorithm by Hendriks & Singer.
- The Liouvilian solution is a generalization of the (q-)hypergeometric solution. Let H is the set of all (q-)hypergeometric sequences and L is the smallest subring of the ring S of all sequences which contains H and is closed under (q-)shifts, summation and interlacing. The elements of L are called Liouvillian sequences and ***reqn*** has a Liouvillian solution if it has a nonzero solution in L.
- The interlacing of m sequences $a_1 = a_{1,\,0}, a_{1,\,1}, \ldots,\ a_2 = a_{2,\,0}, a_{2,\,1}, \ldots,\ \ a_m = a_{m,\,0}, a_{m,\,1}, \ldots$ is the sequence $a_{1,\,0}, a_{2,\,0}, \ldots, a_{m,\,0}, a_{1,\,1}, a_{2,\,1}, \ldots, a_{m,\,1}, \ldots$.
- The solution is constructed by factoring the given equation, finding solutions for the factors and combining the bases ***B[i]*** of the factor's solutions in the solution of the given system.
- The procedure returns the solution that that is a linear combination of the independent solutions. The independent solutions will have indexed coefficients of the form $\_C_1, \_C_2, \ldots, \_C_n$, where $\_C$ is the global name.

Options

- **'indices'**=[n,k] Specifies base names for dummy variables. The default values are the global names $\_n$ and $\_k$, respectively. The $n1$, $n2$, etc. used as summation indices in the Liouvilian solution. The name k is used as the product index.

- If the option **'low'=*lb*** is given, then ***lb*** is used to assign a value of the low bound of the index starting form which the found solution is valid.

Algorithm

- The algorithm by Hendriks & Singer implemented in LiouvillianSolution reduces the task of finding the Liovillian solution of the general form to one of finding the Liouvillian solutions which are the interlacings of the (q-)hypergeometric solutions for the factors of the given systems, which is in turn is reduced to one of finding the (q-)hypergeometric solutions of a specially constructed equation for a factor.
- Combination of the bases ***B[i]*** is using Casaratian determinant. The implementation computes it using the recurrence for the determinant, rather than computing it by the definition. It allows computing more compact expressions in solutions. In addition the determinant's roots are used for defining the lower summation bounds.

Examples

```
> read "LiouvillianSolution.mm";
> rec := y(k)-(1+k-3*k^3)*y(k+2):
> Res := LiouvillianSolution(rec, y(k));
```

$$Res := \begin{cases} (-1)^{\frac{1}{2}k} \prod_{\_k=0}^{\frac{1}{2}k-1} \left(\frac{1}{-1-2\_k+24\_k^3} \right) \_C_1 & irem(k,2)=0 \\ (-1)^{\frac{1}{2}k-\frac{1}{2}} \prod_{\_k=0}^{\frac{1}{2}k-\frac{3}{2}} \left(\frac{1}{1+16\_k+24\_k^3+36\_k^2} \right) \_C_2 & irem(k,2)=1 \end{cases} \quad (7.1)$$

```
> qrec := -(a^2+b*x)*q*y(x)+(-a+a*q)*y(q*x)+y(q^2*x):
> LiouvillianSolution(qrec, y(x));
```

$$\begin{cases} \prod_{\_k0=0}^{\frac{1}{2}\_n-1} \left(a^2+b\,q^{2\_k0}\right) \_C_1 + \prod_{\_k0=0}^{\frac{1}{2}\_n-1} \left(a^2+q\,b\,q^{2\_k0}\right) \_C_2 & irem(\_n,2)=0 \\ \prod_{\_k0=0}^{\frac{1}{2}\_n-\frac{3}{2}} \left(a^2+b\,q^{2\_k0+1}\right) \_C_2\, a + \frac{\prod_{\_k0=0}^{\frac{1}{2}\_n-\frac{3}{2}} \left(a^2+q\,b\,q^{2\_k0+1}\right) \left(a^2+b\right) \_C_1}{a} & irem(\_n,2)=1 \end{cases} \quad (7.$$

```
> qrec := (x+4)*(y(x*q^4)+q*y(x*q^3))-(x-1)*(y(x*q)+q*y(x)):
> LiouvillianSolution(qrec, y(x));
```

$$
\_C_1 \left(\sum_{\_n01=1}^{\_n0-1} \frac{\begin{cases} \displaystyle\prod_{\_k1=1}^{\frac{1}{3}\_n01-1} \left(\frac{q^{3\_k1}-1}{q^{3\_k1}+4} \right) & irem(\_n01,3)=0 \\ 0 & irem(\_n01,3)=1 \\ 0 & irem(\_n01,3)=2 \end{cases}}{(-q)^{-\_n01+1}} \right) (-q)^{-\_n0} + \_C_2 \left(\sum_{\_n01=1}^{\_n0-1} \frac{\begin{cases} 0 & irem(\_n01,3)=0 \\ \displaystyle\prod_{\_k1=0}^{\frac{1}{3}\_n01-\frac{4}{3}} \left(\frac{q^{3\_k1+1}-1}{q^{3\_k1+1}+4} \right) & irem(\_n01,3)=1 \\ 0 & irem(\_n01,3)=2 \end{cases}}{(-q)^{-\_n01+1}} \right) (-q)^{-\_n0} + \_C_3 \left(\sum_{\_n01=1}^{\_n0-1} \frac{\begin{cases} 0 & irem(\_n01,3)=0 \\ 0 & irem(\_n01,3)=1 \\ \displaystyle\prod_{\_k1=0}^{\frac{1}{3}\_n01-\frac{5}{3}} \left(\frac{q^{3\_k1+2}-1}{q^{3\_k1+2}+4} \right) & irem(\_n01,3)=2 \end{cases}}{(-q)^{-\_n01+1}} \right) (-q)^{-\_n0} + \_C_4 (-q)^{-\_n0} \tag{7.3}
$$

```
> rec := (n+1)*n*(n-1)*y(n+4)-n^2*(2*n+3)*(n-1)*y(n+3)+(n-1)*
  (n^4-8*n^2-9*n-1)*y(n+2)+(n+1)*(2*n^4+4*n^3-n^2-5*n-1)*y(n+1)
  -n^3*(n+1)*(n^2-2)*y(n):
```

```
> LiouvillianSolution(rec,y(n));
```

$$
\_C_1 \left(\left(\sum_{n1=2}^{n-1} \left(\right. \right. \right. \tag{7.4}
$$

$$-\frac{\left(\begin{cases}\frac{1}{2}\,2^{\frac{1}{2}n1+\frac{1}{2}}\,\Gamma\left(\frac{1}{2}n1+\frac{1}{2}\right) & irem(n1+1,2)=0\\ 2^{\frac{1}{2}n1}\,\Gamma\left(\frac{1}{2}n1+1\right) & irem(n1+1,2)=1\end{cases}\right)\Gamma\left(n1+\sqrt{2}\right)}{\Gamma\left(\sqrt{2}\right)(-1)^{n1+1}\,\Gamma(n1+1)\,n1}\Bigg)\Bigg)\Bigg($$

$$\begin{cases}\dfrac{2^{\frac{1}{2}n}\,\Gamma\left(\frac{1}{2}n+\frac{1}{2}\right)}{\sqrt{\pi}} & irem(n,2)=0\\ \dfrac{2^{\frac{1}{2}n-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}n\right)}{\sqrt{\pi}} & irem(n,2)=1\end{cases}+\Bigg(\sum_{n1=2}^{n-1}$$

$$\frac{\left(\begin{cases}\dfrac{2^{\frac{1}{2}n1+\frac{1}{2}}\,\Gamma\left(\frac{1}{2}n1+1\right)}{\sqrt{\pi}} & irem(n1+1,2)=0\\ \dfrac{2^{\frac{1}{2}n1}\,\Gamma\left(\frac{1}{2}n1+\frac{1}{2}\right)}{\sqrt{\pi}} & irem(n1+1,2)=1\end{cases}\right)\Gamma\left(n1+\sqrt{2}\right)}{\Gamma\left(\sqrt{2}\right)(-1)^{n1+1}\,\Gamma(n1+1)\,n1}\Bigg)\Bigg($$

$$\left(\left\{\begin{array}{ll} \frac{1}{2}\, 2^{\frac{1}{2} n}\, \Gamma\left(\frac{1}{2} n\right) & irem(n, 2) = 0 \\ 2^{\frac{1}{2} n - \frac{1}{2}}\, \Gamma\left(\frac{1}{2} n + \frac{1}{2}\right) & irem(n, 2) = 1 \end{array}\right.\right) + \_C_2 \left(\left(\sum_{n1=2}^{n-1} \left(\right.\right.\right.$$

$$- \frac{\left(\left\{\begin{array}{ll} \frac{1}{2}\, 2^{\frac{1}{2} n1 + \frac{1}{2}}\, \Gamma\left(\frac{1}{2} n1 + \frac{1}{2}\right) & irem(n1 + 1, 2) = 0 \\ 2^{\frac{1}{2} n1}\, \Gamma\left(\frac{1}{2} n1 + 1\right) & irem(n1 + 1, 2) = 1 \end{array}\right.\right) \Gamma\left(n1 - \sqrt{2}\right)}{\Gamma\left(-\sqrt{2}\right) (-1)^{n1 + 1}\, \Gamma(n1 + 1)\, n1} \left.\left.\right)\right) \left(\right.$$

$$\left\{\begin{array}{ll} \frac{2^{\frac{1}{2} n}\, \Gamma\left(\frac{1}{2} n + \frac{1}{2}\right)}{\sqrt{\pi}} & irem(n, 2) = 0 \\ \frac{2^{\frac{1}{2} n - \frac{1}{2}}\, \Gamma\left(\frac{1}{2} n\right)}{\sqrt{\pi}} & irem(n, 2) = 1 \end{array}\right. + \left(\sum_{n1=2}^{n-1} \right.$$

$$\left(\frac{\left(\begin{cases}\dfrac{2^{\frac{1}{2}\,n1+\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\,n1+1\right)}{\sqrt{\pi}} & irem(n1+1,2)=0\\[2ex] \dfrac{2^{\frac{1}{2}\,n1}\,\Gamma\left(\frac{1}{2}\,n1+\frac{1}{2}\right)}{\sqrt{\pi}} & irem(n1+1,2)=1\end{cases}\right)\Gamma\left(n1-\sqrt{2}\right)}{\Gamma\left(-\sqrt{2}\right)\,(-1)^{n1+1}\,\Gamma(n1+1)\,n1}\right)\Bigg($$

$$\left.\begin{cases}\frac{1}{2}\,2^{\frac{1}{2}\,n}\,\Gamma\left(\frac{1}{2}\,n\right) & irem(n,2)=0\\[1ex] 2^{\frac{1}{2}\,n-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\,n+\frac{1}{2}\right) & irem(n,2)=1\end{cases}\right)\Bigg)+\_C_3\Bigg($$

$$\begin{cases}\dfrac{2^{\frac{1}{2}\,n}\,\Gamma\left(\frac{1}{2}\,n+\frac{1}{2}\right)}{\sqrt{\pi}} & irem(n,2)=0\\[2ex] \dfrac{2^{\frac{1}{2}\,n-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\,n\right)}{\sqrt{\pi}} & irem(n,2)=1\end{cases}\Bigg)+\_C_4\Bigg($$

$$\begin{cases}\frac{1}{2}\,2^{\frac{1}{2}\,n}\,\Gamma\left(\frac{1}{2}\,n\right) & irem(n,2)=0\\[1ex] 2^{\frac{1}{2}\,n-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\,n+\frac{1}{2}\right) & irem(n,2)=1\end{cases}\Bigg)$$

See Also

LREtools, LREtools[hypergeomsols], piecewise, QDifferenceEquations, QDifferenceEquations

[QHypergeometricSolution]

References

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