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Existence of Liouvillian solutions in the problem of motion of a heavy gyrostat under the action of gyroscopic forces

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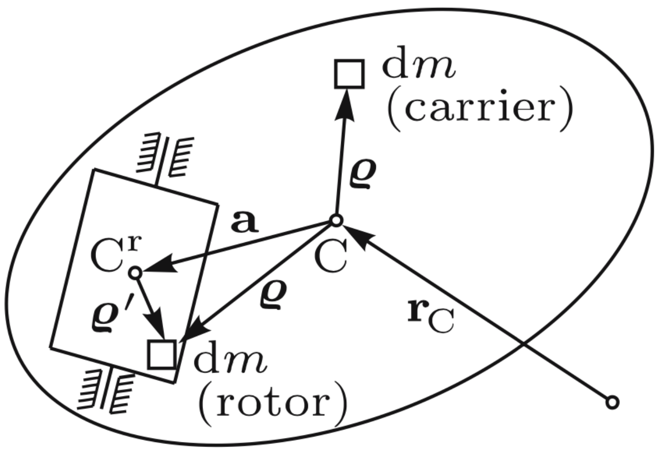
Definition of a Gyrostat - I



A gyrostat is a mechanical system consisting of several bodies which possesses the rigid body property, which means that the mass distribution of the system does not change over time.

In the simplest case, the gyrostat consists of two bodies. Let us consider a heavy rigid body with a fixed point (the carrier). Suppose that there is an axis associated with the body around which the rotor can rotate.

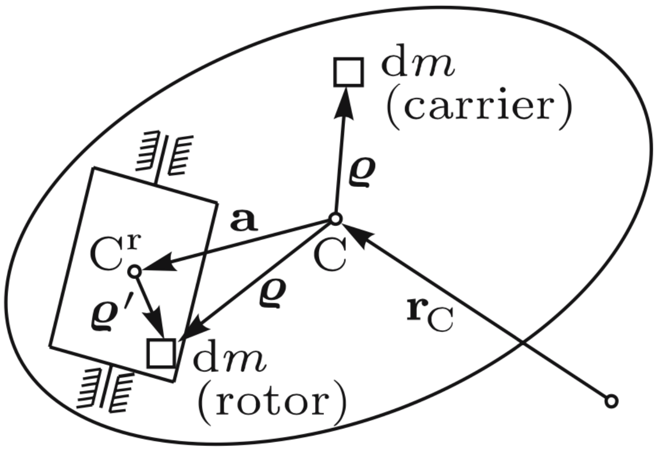
The angular momentum of the system can be written as

$$\mathbf{K}_O = \mathbf{J}_O\boldsymbol{\omega} + \mathbf{J}^r\boldsymbol{\omega}_r$$

In this expression $\mathbf{J}_O$ is the inertia matrix of the carrier with respect to the fixed point; $\boldsymbol{\omega}$ is the angular velocity vector of the carrier; $\mathbf{J}^r$ is the inertia matrix of the rotor with respect to its center of mass; $\boldsymbol{\omega}_r$ is the angular velocity vector of the rotor with respect to the carrier.

Both inertia matrices have constant components in the coordinate frame which is rigidly connected with the carrier. Vector $\mathbf{J}^r\boldsymbol{\omega}_r$ is the angular momentum of motion of the rotor with respect to the carrier. We shall denote it as $\mathbf{s}$.

Definition of a Gyrostat - II



Theorem of Angular Momentum Change states that

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \dot{\mathbf{s}} + \left[\boldsymbol{\omega} \times \left(\mathbf{J}\boldsymbol{\omega} + \mathbf{s}\right)\right] = \mathbf{M}_o$$

$\mathbf{M}_o$ is the torque of external forces with respect to the fixed point.

In the simplest case the angular velocity of rotor is constant. Suppose that a gyrostat is under the effect of gravity and gyroscopic forces. Therefore, the equation takes the form

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \left[\boldsymbol{\omega} \times \left(\mathbf{J}\boldsymbol{\omega} + \mathbf{s}\right)\right] = Mg\left[\boldsymbol{\gamma} \times \mathbf{r}_G\right] + \left[S\boldsymbol{\gamma} \times \boldsymbol{\omega}\right]$$

S is a symmetric matrix: $S = S^T$.

Together with the Poisson's equation $\dot{\boldsymbol{\gamma}} + \left[\boldsymbol{\omega} \times \boldsymbol{\gamma}\right] = 0$ we obtain a closed system of equations of motion of the gyrostat.

Euler-Poisson Equations - I

$$A_1\dot{\omega}_1 + (A_3 - A_2)\omega_2\omega_3 + s_3\omega_2 - s_2\omega_3 = Mg(x_3\gamma_2 - x_2\gamma_3) + \omega_3(S\boldsymbol{\gamma})_2 - \omega_2(S\boldsymbol{\gamma})_3,$$
$$A_2\dot{\omega}_2 + (A_1 - A_3)\omega_1\omega_3 + s_1\omega_3 - s_3\omega_1 = Mg(x_1\gamma_3 - x_3\gamma_1) + \omega_1(S\boldsymbol{\gamma})_3 - \omega_3(S\boldsymbol{\gamma})_1,$$
$$A_3\dot{\omega}_3 + (A_2 - A_1)\omega_1\omega_2 + s_2\omega_1 - s_1\omega_2 = Mg(x_2\gamma_1 - x_1\gamma_2) + \omega_2(S\boldsymbol{\gamma})_1 - \omega_1(S\boldsymbol{\gamma})_2;$$
$$\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3,\ \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1,\ \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2.$$

For any values of parameters of the system and for any initial conditions we have three first integrals of the Euler – Poisson equations.

The energy integral

$$H = \frac{1}{2}\left(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2\right) + Mg(x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3) = E = \text{const}$$

The area integral

$$K = (A_1\omega_1 + s_1)\gamma_1 + (A_2\omega_2 + s_2)\gamma_2 + (A_3\omega_3 + s_3)\gamma_3 + \frac{1}{2}(S\boldsymbol{\gamma}\times\boldsymbol{\gamma}) = k = \text{const}$$

The geometrical integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

Thus, for the integrability of the Euler – Poisson equations, we need to find only one additional autonomous first integral. In general case the additional integral does not exist. However, for the special initial conditions we can find the additional special integral.

Euler-Poisson Equations - II

Theorem. (Kosov A.A.) Suppose that following conditions are true:
1) $x_3 = 0, \quad A_2\left(A_3 - A_1\right)x_2^2 = A_1\left(A_2 - A_3\right)x_1^2, \quad s_3 = 0, \quad A_2 \ge A_3 \ge A_1;$
2) Matrix S has form

$$S = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

where $s_{11}x_2 - s_{12}x_1 = 0,\ s_{12}x_2 - s_{22}x_1 = 0.$

Therefore, Euler-Poisson equations admit the additional special integral of the form

$$A_1\omega_1 x_1 + A_2\omega_2 x_2 + \frac{A_1 x_1\left(s_2 x_1 - s_1 x_2\right)}{\left(A_3 - A_1\right)x_2} = 0.$$

If we take conditions of the theorem into consideration, then the system of equations of motion will take the following form:

Euler-Poisson Equations - III

$$A_1\dot{\omega}_1+\left(A_3-A_2\right)\omega_2\omega_3-s_2\omega_3=-Mgx_2\gamma_3+\lambda x_2\omega_3\left(\gamma_1 x_1+\gamma_2 x_2\right),$$
$$A_2\dot{\omega}_2+\left(A_1-A_3\right)\omega_1\omega_3+s_1\omega_3=Mgx_1\gamma_3-\lambda x_1\omega_3\left(\gamma_1 x_1+\gamma_2 x_2\right),$$
$$A_3\dot{\omega}_3+\left(A_2-A_1\right)\omega_1\omega_2+s_2\omega_1-s_1\omega_2=Mg\left(x_2\gamma_1-x_1\gamma_2\right)+\lambda\left(\omega_2 x_1-\omega_1 x_2\right)\left(\gamma_1 x_1+\gamma_2 x_2\right);$$
$$\dot{\gamma}_1=\omega_3\gamma_2-\omega_2\gamma_3,\;\; \dot{\gamma}_2=\omega_1\gamma_3-\omega_3\gamma_1,\;\; \dot{\gamma}_3=\omega_2\gamma_1-\omega_1\gamma_2.$$

Here, λ is an auxiliary multiplier.

First integrals of the system can be rewritten as follows

$$H=\frac{1}{2}\left(A_1\omega_1^2+A_2\omega_2^2+A_3\omega_3^2\right)+Mg\left(x_1\gamma_1+x_2\gamma_2\right)=E=\text{const},$$
$$K=\left(A_1\omega_1+s_1\right)\gamma_1+\left(A_2\omega_2+s_2\right)\gamma_2+A_3\omega_3\gamma_3+\frac{\lambda}{2}\left(x_1\gamma_1+x_2\gamma_2\right)^2=k=\text{const},$$
$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1.$$

To prove the existence of the additional integral

$$A_1\omega_1 x_1+A_2\omega_2 x_2+\frac{A_1 x_1\left(s_2 x_1-s_1 x_2\right)}{\left(A_3-A_1\right)x_2}=0,$$

we shall rewrite our system using special Kharlamov axes and introduce new variables.

Euler-Poisson equations in Kharlamov axes - I

The transition from principal axes of inertia of the body in a fixed point to special Kharlamov axes is defined by following formulas

$$\mathbf{e}_{\mathrm{I}} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\mathbf{e}_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\mathbf{e}_2, \quad \mathbf{e}_{\mathrm{II}} = -\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\mathbf{e}_1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\mathbf{e}_2, \quad \mathbf{e}_{\mathrm{III}} = \mathbf{e}_3.$$

We denote components of the angular momentum relative to the fixed point projected onto special axes as L_1, L_2, L_3. We also introduce components ν_1, ν_2, ν_3 of vector $\boldsymbol{\gamma}$ projected onto special axes. Therefore

$$L_1 = \frac{A_1\omega_1 x_1 + A_2\omega_2 x_2}{\sqrt{x_1^2 + x_2^2}}, \quad L_2 = \frac{A_2\omega_2 x_1 - A_1\omega_1 x_2}{\sqrt{x_1^2 + x_2^2}}, \quad L_3 = A_3\omega_3,$$

$$\nu_1 = \frac{\gamma_1 x_1 + \gamma_2 x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \nu_2 = \frac{\gamma_2 x_1 - \gamma_1 x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \nu_3 = \gamma_3, \quad k_1 = \frac{s_1 x_1 + s_2 x_2}{\sqrt{x_1^2 + x_2^2}},$$

$$k_2 = \frac{s_2 x_1 - s_1 x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \Gamma = Mg\sqrt{x_1^2 + x_2^2}, \quad c = \frac{1}{A_3},$$

$$a = \frac{A_2 x_1^2 + A_1 x_2^2}{A_1 A_2\left(x_1^2 + x_2^2\right)}, \quad b = \frac{\left(A_1 - A_2\right) x_1 x_2}{A_1 A_2\left(x_1^2 + x_2^2\right)}, \quad \Lambda = \lambda\left(x_1^2 + x_2^2\right).$$

Euler-Poisson equations in Kharlamov axes - II

Using new variables, the system can be rewritten as follows:

$$\dot{L}_1 = -bL_3\left(L_1 - \frac{ck_2}{b}\right),$$
$$\dot{L}_2 = (a-c)L_1L_3 + bL_2L_3 - ck_1L_3 - c\Lambda L_3\nu_1 + \Gamma\nu_3,$$
$$\dot{L}_3 = -(a-c)L_1L_2 + bL_1^2 - bL_2^2 + (k_1b - k_2a)L_1 + (k_1c - k_2b)L_2 + b\Lambda L_1\nu_1 + c\Lambda L_2\nu_1 - \Gamma\nu_2,$$

$$\dot{\nu}_1 = cL_3\nu_2 - (cL_2 + bL_1)\nu_3,$$
$$\dot{\nu}_2 = (aL_1 + bL_2)\nu_3 - cL_3\nu_1,$$
$$\dot{\nu}_3 = (bL_1 + cL_2)\nu_1 - (aL_1 + bL_2)\nu_2.$$

The additional special integral of the system can be found directly from the first equation. It has the form

$$L_1 \equiv \frac{ck_2}{b}.$$

Euler-Poisson equations in Kharlamov axes - III

Under these condition equations are simplified and take the form

$$\dot{\tilde{L}}_2 = b\tilde{L}_2 L_3 + \left(F - Gc\right) L_3 - c\Lambda L_3 \nu_1 + \Gamma \nu_3, \quad \dot{L}_3 = -b\tilde{L}_2^2 - \left(F - Gc\right)\tilde{L}_2 + c\Lambda \tilde{L}_2 \nu_1 - \Gamma \nu_2,$$

$$\dot{\nu}_1 = cL_3\nu_2 - c\tilde{L}_2\nu_3, \quad \dot{\nu}_2 = -cL_3\nu_1 + b\tilde{L}_2\nu_3 + F\nu_3, \quad \dot{\nu}_3 = c\tilde{L}_2\nu_1 - b\tilde{L}_2\nu_2 - F\nu_2.$$

Here we introduce the following notations:

$$\frac{\left(ac - b^2\right)}{b} k_2 = F, \quad \frac{ck_2}{b} + k_1 = G, \quad L_2 + k_2 = \tilde{L}_2.$$

This system possesses following first integrals

$$\frac{c}{2}\left(\tilde{L}_2^2 + L_3^2\right) + \Gamma \nu_1 = E; \quad G\nu_1 + \tilde{L}_2\nu_2 + L_3\nu_3 + \frac{\Lambda}{2}\nu_1^2 = k; \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Euler-Poisson equations in Kharlamov axes - IV

Let us introduce now the dimensionless components of the angular momentum, the dimensionless time, the dimensionless constants of the first integrals:

$$\tilde{L}_2 = y\sqrt{\frac{\Gamma}{c}}, \quad L_3 = z\sqrt{\frac{\Gamma}{c}}, \quad t = \frac{\tau}{\sqrt{\Gamma c}}, \quad h = \frac{E}{\Gamma}, \quad p_1 = k\sqrt{\frac{c}{\Gamma}},$$

and the dimensionless parameters:

$$d_1 = \frac{b}{c}, \quad A = \frac{F}{\sqrt{\Gamma c}}, \quad B = G\sqrt{\frac{c}{\Gamma}}, \quad Q = \Lambda\sqrt{\frac{c}{\Gamma}}.$$

Now we can rewrite equations in dimensionless form

$$\frac{dy}{d\tau} = d_1 yz + (A - B)z - Qz\nu_1 + \nu_3, \quad \frac{dz}{d\tau} = -d_1 y^2 - (A - B)y + Qy\nu_1 - \nu_2,$$

$$\frac{d\nu_1}{d\tau} = z\nu_2 - y\nu_3, \quad \frac{d\nu_2}{d\tau} = d_1 y\nu_3 - z\nu_1 + A\nu_3, \quad \frac{d\nu_3}{d\tau} = -d_1 y\nu_2 + y\nu_1 - A\nu_2.$$

This system possesses following first integrals

$$\frac{(y^2 + z^2)}{2} + \nu_1 = h; \quad y\nu_2 + z\nu_3 + B\nu_1 + \frac{Q}{2}\nu_1^2 = p_1; \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Euler-Poisson equations in Kharlamov axes - V

We introduce now the polar coordinates by putting

$$y = x\cos\varphi, \quad z = x\sin\varphi.$$

From the first integral we have the following equation:

$$\nu_1 = \nu_1(x) = h - \frac{x^2}{2}.$$

From the trivial identity

$$\left(\nu_2^2 + \nu_3^2\right)\left(y^2 + z^2\right) = \left(y\nu_2 + z\nu_3\right)^2 + \left(z\nu_2 - y\nu_3\right)^2$$

we obtain:

$$x^2\left(1 - \left(h - \frac{x^2}{2}\right)^2\right) = \left(p_1 - B\left(h - \frac{x^2}{2}\right) - \frac{Q}{2}\left(h - \frac{x^2}{2}\right)^2\right)^2 + \left(z\nu_2 - y\nu_3\right)^2.$$

We will take that

$$z\nu_2 - y\nu_3 = \pm\sqrt{x^2\left(1 - \left(h - \frac{x^2}{2}\right)^2\right) - \left(p_1 - B\left(h - \frac{x^2}{2}\right) - \frac{Q}{2}\left(h - \frac{x^2}{2}\right)^2\right)^2}.$$

These equation (combined with the second integral) allows us to find expressions for $\nu_2 = \nu_2(y, z, \nu_1)$ and $\nu_3 = \nu_3(y, z, \nu_1)$.

Obtaining the second order linear equation - I

For the variables x and φ we have the following system of differential equations

$$x\frac{dx}{d\tau}=-\sqrt{x^2\left(1-\left(\frac{x^2}{2}-h\right)^2\right)-\left(p_1-B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2\right)^2},$$

$$x^2\frac{d\varphi}{d\tau}=-d_1x^3\cos\varphi-\left(A-B\right)x^2+Qx^2\left(h-\frac{x^2}{2}\right)-p_1+B\left(h-\frac{x^2}{2}\right)+\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2.$$

From this system we obtain the equation for the function $\varphi=\varphi(x)$:

$$\frac{d\varphi}{dx}=\frac{d_1x^3\cos\varphi+\frac{3}{8}Qx^4+\left(A-\frac{B}{2}-\frac{Qh}{2}\right)x^2+\left(p_1-Bh-\frac{Qh^2}{2}\right)}{x\sqrt{x^2\left(1-\left(\frac{x^2}{2}-h\right)^2\right)-\left(p_1-B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2\right)^2}}.$$

Using the change of variable

$$w=\tan\frac{\varphi}{2}$$

we reduce this equation to Riccati equation.

Obtaining the second order linear equation - II

Riccati equation for function $w=w(x)$ has form

$$\frac{dw}{dx}=\frac{-d_1x^3+\frac{3}{8}Qx^4+\left(A-\frac{B}{2}-\frac{Qh}{2}\right)x^2+\left(p_1-Bh-\frac{Qh^2}{2}\right)}{2x\sqrt{x^2\left(1-\left(\frac{x^2}{2}-h\right)^2\right)-\left(p_1-B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2\right)^2}}w^2+$$

$$+\frac{d_1x^3+\frac{3}{8}Qx^4+\left(A-\frac{B}{2}-\frac{Qh}{2}\right)x^2+\left(p_1-Bh-\frac{Qh^2}{2}\right)}{2x\sqrt{x^2\left(1-\left(\frac{x^2}{2}-h\right)^2\right)-\left(p_1-B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2\right)^2}}$$

It is well-known from the general theory of ODE that if Riccati equation has the form

$$\frac{dw}{dx}=f_2(x)w^2+f_1(x)w+f_0(x),$$

then the substitution of the form

$$u(x)=\exp\left(-\int f_2(x)w(x)dx\right)$$

reduces it to the second order linear differential equation

$$\frac{d^2u}{dx^2}-\left[\frac{1}{f_2}\frac{df_2}{dx}+f_1\right]\frac{du}{dx}+f_0f_2u=0.$$

Obtaining the second order linear equation - III

To sum up, the second order linear differential equation has the form

$$\frac{d^2u}{dx^2}+a(x)\frac{du}{dx}+b(x)u=0,$$

$$a(x)=\frac{P_{12}(x)}{xP_4(x)P_8(x)},\quad b(x)=-\frac{P_4(x)\left(P_4(x)+16d_1x^3\right)}{4x^2P_8(x)},$$

$$P_{12}(x)=3Q^3x^{12}-16Q^2d_1x^{11}+24\left(A-\frac{B}{2}-\frac{Qh}{2}\right)Q^2x^{10}+64d_1\left(Q^2h+BQ^2-2\right)x^9+$$

$$+\left(-28h^2Q^3+\left((-128A-56B)Q^2+64Q\right)h+88Q^2p_1+(-128AB+16B^2)Q^2+256A-128B\right)x^8$$

$$+(224Q^3h^3+\left((192A+678B)Q^2-384Q\right)h^2+\left(-576Q^2p_1+384B(A+B)Q^2-512A-256B\right)h+$$

$$+\left(-128Ap_1-576Bp_1+384\right)Q+128AB^2-64B^3+512p_1)x^6-$$

$$-512\left(\frac{Q^2h^3}{2}+\left(\frac{3BQ^2}{2}-1\right)h^2+\left(B^2-Q^2p_1\right)h-Bp_1+1\right)d_1x^5$$

$$-384\left(\frac{1}{2}Qh^2+Bh-p_1\right)\left(\frac{9Q^2h^2}{4}+\left(\frac{9BQ}{2}-4\right)h+B^2-\frac{5Qp_1}{2}\right)x^4+1024d_1\left(\frac{1}{2}Qh^2+Bh-p_1\right)^2x^3-$$

$$-512\left(-\frac{5Q^2h^3}{4}+\left(2+\left(\frac{A}{2}-\frac{15B}{4}\right)Q\right)h^2+\left(\frac{5Qp_1}{2}+B\left(A-\frac{5B}{2}\right)\right)h-Ap_1+\frac{5Bp_1}{2}-2\right)\bullet$$

$$\bullet\left(\frac{1}{2}Qh^2+Bh-p_1\right)x^2-512\left(\frac{1}{2}Qh^2+Bh-p_1\right)^2,$$

Obtaining the second order linear equation - IV

$$P_8(x) = Q^2x^8 - 8\left(BQ + Q^2h - 2\right)x^6 + 8\left(3Q^2h^2 + \left(6BQ - 8\right)h + 2B^2 - Qp_1\right) +$$

$$+32\left(-\frac{Q^2h^3}{2} + h^2\left(1 - \frac{3BQ}{2}\right) - \left(B^2 - Qp_1\right)h + Bp_1 - 1\right)x^2 + 64\left(Bh - p_1 + \frac{1}{2}Qh^2\right)^2.$$

$$P_4(x) = 3Qx^4 - 8d_1x^3 + \left(8A - 4B - 4Qh\right)x^2 + \left(8p_1 - 8Bh - 4Qh^2\right).$$

<u>Theorem 1.</u> The solution of the problem of motion of a heavy gyrostat with a fixed point under the action of gyroscopic forces in the Hess case of integrability can be reduced to solving the second order linear differential equation with rational coefficients.

Therefore, we can use the Kovacic algorithm to find liouvillian solutions of the differential equation. We implemented this algorithm using Maple.

General solution

If we manage to find explicitly the general solution of the linear second order differential equation

$$u = C_1 u_1(x) + C_2 u_2(x),$$

then we can find the expression for $\varphi = \varphi(x),$ from which we can find explicit expression for variables $y = y(x),\ z = z(x).$

Now, using the following system of equations

$$y(x)\nu_3 - z(x)\nu_2 = -\sqrt{x^2\left(1-\left(\frac{x^2}{2}-h\right)^2\right)-\left(p_1 - B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2\right)^2},$$

$$y(x)\nu_2 + z(x)\nu_3 = p_1 - B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2,$$

we can find the expressions for $\nu_2 = \nu_2(x),\ \nu_3 = \nu_3(x).$ The form of the expression $\nu_1 = \nu_1(x)$ can be found from the first integral

$$\nu_1(x) = h - \frac{x^2}{2},$$

the expression for variable $x = x(\tau)$ can be found directly from the equation

$$x\frac{dx}{d\tau} = -\sqrt{x^2\left(1-\left(\frac{x^2}{2}-h\right)^2\right)-\left(p_1 - B\left(h-\frac{x^2}{2}\right)-\frac{Q}{2}\left(h-\frac{x^2}{2}\right)^2\right)^2}$$

Results

The application of the Kovacic algorithm to the problem of motion of a heavy gyrostat with a fixed point under the action of gyroscopic forces gives the following result.

Theorem 2. Let $Q \neq 0$ (gyroscopic forces are present) and $d_1 \neq 0$ (the mass distribution of the gyrostat does not correspond to the Lagrange integrable case). Then the second order linear differential equation admits a general solution expressed in terms of Liouvillian functions under the condition

$$A = \frac{d_1^2 + 1}{Q}.$$

Thank you for your attention!