Quantum Mechanics Through the Lens of Finite Groups: Computer Algebra Insights

6th International Conference “Computer Algebra”
June 23-25, 2025
Moscow, Russia

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June 23, 2025

▸ Hilbert

Quantum evolution

- Schrödinger equation

$$\mathbf{i}\hbar\frac{\partial}{\partial t}\left|\psi_t\right\rangle = H\left|\psi_t\right\rangle \rightsquigarrow \left|\psi_t\right\rangle = U_t\left|\psi_0\right\rangle$$

continuous one-parameter unitary group

$$U_t = \mathrm{e}^{-\mathbf{i}\frac{H}{\hbar}t} = \left(\mathrm{e}^{-\mathbf{i}\frac{H}{\hbar}}\right)^t = \mathtt{E}^t$$

- Without empirical losses, the evolution operator $\mathtt{E}$ can be a generator of a representation of the finite cyclic group $\mathbb{Z}_N$

▸ Banks

$$N \sim \begin{cases} \mathrm{Exp}(\mathrm{Exp}(20)) & \text{for } 1 \text{ cm}^3 \text{ of matter,} \\ \mathrm{Exp}(\mathrm{Exp}(123)) & \text{for the entire Universe.} \end{cases}$$

Finite vs Lie group: $\mathbb{Z}_N$ vs $\mathrm{U}(1)$

- $\mathrm{U}(1) :\approx \mathbb{Z}_N$ for large N
- Chinese remainder theorem implies

$$\mathbb{Z}_N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \quad \text{if } N = n_1 n_2 \text{ and } \gcd(n_1, n_2) = 1$$

$$\Downarrow$$

$$\mathbb{Z}_N \cong \mathbb{Z}_{p_1^{\ell_1}} \times \cdots \times \mathbb{Z}_{p_m^{\ell_m}}$$

  - $N = p_1^{\ell_1} \cdots p_m^{\ell_m}$ is prime factorization of N
  - $\mathbb{Z}_{p^\ell} \xrightarrow{*} \mathbb{F}_{p^\ell}$ is a Galois field — crucial role in quantum mechanics
  - Topologically, $\mathbb{Z}_N$ is a discrete multidimensional torus, resembles the circle $\mathrm{U}(1)$ topology only if N is a prime number

Regular representation of $\mathbb{Z}_N$

Cyclic permutations of the group elements

- Generator

$$X = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \qquad X|_{N=2} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ a Pauli matrix}$$

- Position or ontic ('t Hooft) or computational (quantum informatics) basis

$$\mathrm{B}_X = \{|0\rangle, \dots, |N-1\rangle\}$$

- Position operator in ontic basis

$$\widehat{x} = \sum_{x=0}^{N-1} x\,|x\rangle\langle x| = \mathrm{diag}\,(0, 1, \dots, N-1)$$

- Generator of evolution with velocity v: $X_v = X^v$

$$\widehat{x}_t = X_v^t \widehat{x}_0 X_v^{-t}$$

in components $\quad x_t = x_0 + vt \mod N$

Irreducible decomposition over a splitting field – here, over $\mathbb{Q}(\omega)$, a dense subfield of $\mathbb{C}$

ω is a Nth primitive root of unity, e.g., $\omega = e^{2\pi \mathbf{i}/N}$

- Generator

$$Z = FXF^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix} \qquad Z|_{N=2} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$F = \frac{1}{\sqrt{N}} \left(\omega^{ij} \right)$ is the Fourier transform

- Momentum basis

$$B_Z = \left\{ \left| \widetilde{0} \right\rangle, \left| \widetilde{1} \right\rangle, \ldots, \left| \widetilde{N-1} \right\rangle \right\}$$

- Momentum operator in momentum basis

$$\widehat{p} = \sum_{p=0}^{N-1} p \, |\widetilde{p}\rangle\langle\widetilde{p}| = \operatorname{diag}(0, 1, \ldots, N-1)$$

Hamiltonian $\widehat{H} = \widehat{p}/N$ (cf. $E = pc$, energy-momentum relation for photon)

Interplay between X and Z leads to quantum effects

- Bases B_X and B_Z are mutually unbiased (Bohr's complementarity)

$$\left|\left\langle \widetilde{\ell} \middle| k \right\rangle\right|^2 = \frac{1}{N}$$

- X, Z generate a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ on Hilbert space $\mathcal{H}_N$

- Direct calculation $\rightsquigarrow$ $ZX = \omega XZ$, the Weyl commutation relation — a refinement of the non-physical Heisenberg canonical commutation relation $[\widehat{x}, \widehat{p}] = \mathbf{i}\hbar$

▸ Weyl

Finite groups acting on $\mathcal{H}_N$: Weyl–Schwinger legacy and beyond

- Generators: $\tau = -\,\mathrm{e}^{\pi\mathbf{i}/N}$, X, Z, F,

  $S = \operatorname{diag}\left(\tau^{i(i+N)}\right)$ is unitary image of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{Z})$

  - Roots of unity: $\mathbb{K}_{\overline{N}} = \langle\tau\rangle$, where $\overline{N} = \begin{cases} N, & N = 2k+1, \\ 2N, & N = 2k\,. \end{cases}$

- Weyl–Heisenberg group: $\mathrm{WH}(N) = \langle \tau, X, Z\rangle$, size $= N^3$ or $2N^3$
  - Displacement operators $\mathrm{D}_{(p_1,p_2)} = \tau^{p_1p_2}X^{p_1}Z^{p_2}$ form projective Weyl–Heisenberg group $\mathrm{PWH}(N) = \mathrm{WH}(N)\,/\mathbb{K}_{\overline{N}} \cong \mathbb{Z}_N \times \mathbb{Z}_N$ describing quantum evolutions
  - Parameters $(p_1,p_2) \in \mathbb{Z}^2$ form finite phase space $\mathrm{T}^2 = \mathbb{Z}_N \times \mathbb{Z}_N$ with symplectic symmetry group $\mathrm{Sp}(2,\mathbb{Z}_N) \cong \mathrm{SL}(2,\mathbb{Z}_N)$, the group of outer automorphisms of $\mathrm{WH}(N)$

- Clifford group: $\mathrm{CL}(N) = \langle X, F, S\rangle$.
  - $\mathrm{CL}(N)$ is the group of symmetries of $\mathrm{WH}(N)$ combining both inner and outer automorphisms: $\mathrm{CL}(N) = \mathrm{Aut}(\mathrm{WH}(N)) \cong \mathrm{WH}(N) \rtimes \mathrm{Sp}(2,\mathbb{Z}_N)$
  - Projective Clifford group: $\mathrm{PCL}(N) = \mathrm{CL}(N)\,/\mathcal{Z}(\mathrm{CL}(N))$

OrdinaryQM

Decomposition of quantum systems: continuous vs finite group

$$\mathcal{H}_N = \mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2} \otimes \cdots \otimes \mathcal{H}_{n_m}, \quad N = n_1 \cdot n_2 \cdots n_m, \quad \gcd(n_i, n_j) = 1$$

- Continuous unitary groups

$$\mathsf{U}(N)\,\mathcal{H}_N = \mathsf{U}(n_1)\,\mathcal{H}_{n_1} \otimes \cdots \otimes \mathsf{U}(n_m)\,\mathcal{H}_{n_m}$$

$$\Downarrow \quad \text{a bit of tensor algebra}$$

$$\mathsf{U}(N)\,\mathcal{H}_N = \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_m}$$

$$\underline{\mathsf{U}(N) > \mathsf{U}(n_1) \otimes \cdots \otimes \mathsf{U}(n_m)}$$

- Clifford groups

$$\Big(\overbrace{\mathrm{CL}(n_1) \otimes \cdots \otimes \mathrm{CL}(n_m)}^{\mathrm{CL}(N)}\Big)\mathcal{H}_N = \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_m}$$

$$\underline{\mathrm{CL}(N) \equiv \mathrm{CL}(n_1) \otimes \cdots \otimes \mathrm{CL}(n_m)}$$

no quantum interferences, no entanglement, no energy exchange
between $n_i = p_i^{\ell_i}$ and $n_j = p_j^{\ell_j}$, $\qquad p_i \neq p_j$

Chinese remainder theorem

$$0 \le k < N = n_1 \cdot n_2 \cdots n_m, \quad \gcd(n_i, n_j) = 1$$
$$\Downarrow$$
$$k \leftrightarrow (r_1, r_2, \ldots, r_m)\,, \quad r_i = k \mod n_i$$

- ring isomorphism $\mathbb{Z}_N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m}$
  isomorphic map $(r_1, r_2, \ldots, r_m) \mapsto k \in \mathbb{Z}_N$

  $$k = \sum_i \underbrace{r_i N_i^{-1}}_{k_i \in \mathbb{Z}_{n_i}} N_i \mod N$$

  $N_i = N/n_i \in \mathbb{Z}_N$
  $N_i^{-1} \in \mathbb{Z}_{n_i}$ is the multiplicative inverse of N_i within $\mathbb{Z}_{n_i}$
- dual map $k \leftrightarrow (k_1, k_2, \ldots, k_m)\,,\ k_i \in \mathbb{Z}_{n_i}$

  $$k = \sum_i k_i N_i \mod N$$
  $$\Downarrow$$
  $$\frac{k}{N} = \sum_i \frac{k_i}{n_i} \mod 1 \ \rightsquigarrow$$ additivity of energy

Additivity of energy in a composite quantum system

$$E(A \cup B) = E(A) + E(B) + \Delta E(A, B)$$

- Planck relation $E = h\nu$, energy = frequency
- Hamiltonian $H = \mathbf{i}\hbar \ln U$

  U is a generator of a $\mathbb{Z}_n$-evolution $\Longrightarrow$ $H \sim \mathrm{diag}\Big(E_{k/n}\Big)$, $E_{k/n} = \dfrac{k}{n}$
- Composite system

$$U_N = U_{n_1} \otimes U_{n_2} \otimes \cdots \otimes U_{n_m}$$
$$\downarrow \ln$$
$$H_N = H_{n_1} \otimes \mathbb{1}_{n_2} \otimes \cdots \otimes \mathbb{1}_{n_m} + \mathbb{1}_{n_1} \otimes H_{n_2} \otimes \cdots \otimes \mathbb{1}_{n_m} + \cdots + \mathbb{1}_{n_1} \otimes \mathbb{1}_{n_2} \otimes \cdots \otimes H_{n_m}$$

  Additivity of energy as dual map in Chinese remainder theorem

$$E_{k/N} = \sum_i E_{k_i/n_i} \Longleftrightarrow \frac{k}{N} = \sum_i \frac{k_i}{n_i} \mod 1$$

Constructive quantum states $\mathrm{CQS}(N)$

- standard QM: complex projective space, homogeneous space of $\mathrm{U}(N)$

$$\mathbb{P}(\mathcal{H}_N) = \mathbb{CP}^{N-1} \cong \mathrm{Orb}_{\mathrm{U}(N)}(|0\rangle) = \mathrm{U}(N)\,|0\rangle$$

- a trial set of $\mathrm{CQS}(N)$
  1. must be $\mathrm{CL}(N)$-invariant:
     $$\mathrm{CQS}(N) = \bigcup_a \mathcal{O}_a, \quad \mathcal{O}_a = \mathrm{Orb}_{\mathrm{CL}(N)}(|a\rangle)$$
  2. must contain ontic vectors: $\mathcal{O}_0 \ni |0\rangle, |1\rangle, \dots, |N-1\rangle$
  3. only rational Born transition probabilities are allowed:
     $$|a\rangle, |b\rangle \in \mathrm{CQS}(N) \implies |\langle a \mid b\rangle|^2 \in \mathbb{Q}$$
  4. phase factors must be elements of the center $\mathcal{Z}(\mathrm{CL}(N))$

Computations in dimensions 2 and 3

Generators, centers, and sizes of $\mathrm{CL}(N)$, $\omega = \exp(2\pi\mathbf{i}/3)$

N	X	F	S	$\mathcal{Z}$	ord
2	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$	$\mathbb{K}_8$	192
3	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\mathbb{K}_{12}$	2592

Distance between states $\mathrm{Dist}(a, b) = 1 - \mathbf{P}(a, b) = \sin^2 \mathrm{D_{FS}}(a, b)$

$\mathbf{P}(a, b) = |\langle a \,|\, b \rangle|^2$ is Born's transition probability,

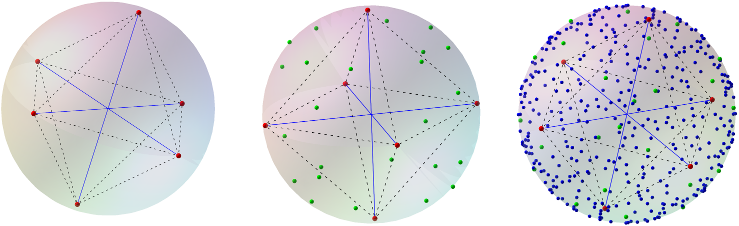
$\mathrm{D_{FS}}(a, b)$ is the Fubini–Study distance in $\mathbb{CP}^{N-1}$

Estimate of density in $\mathbb{CP}^{N-1}$ of the computationally reached set of states $R \subset \mathrm{CQS}(N)$

$$\Delta(R) = \max_{a \in R} \min_{b \in R \setminus \{a\}} \mathrm{Dist}(a, b), \quad R = \mathcal{O}_0 \cup \bigcup_{i=1}^{N_{\mathrm{orb}}-1} \mathcal{O}_i$$

N	N_{orb}	$\lvert R \rvert$	$\Delta(\mathcal{O}_0)$	$\Delta(R)$
2	986	23646	1/2	$1/1515 \approx 10^{-3}$
3	169	27237	2/3	$1/99 \approx 10^{-2}$

Initial steps in generating $\mathrm{CQS}(2)$



(a) (b) (c)

(a) vectors of $\mathcal{O}_0$ form the octahedron vertices, spatial diagonals form complete set of mutually unbiased bases:

$$\mathcal{O}_0 = \left\{ |0\rangle, |1\rangle ; \quad \frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}} ; \quad \frac{|0\rangle+\mathbf{i}|1\rangle}{\sqrt{2}}, \frac{|0\rangle-\mathbf{i}|1\rangle}{\sqrt{2}} \right\}$$

(b) pairwise interferences of the vectors in (a) with rational transition probabilities add one orbit of size 24

(c) pairwise interferences of the vectors in (b) add 16 orbits of size 24

David Hilbert

David Hilbert. *On the infinite*

"Our principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought — a remarkable harmony between being and thought."

To 1

Gerard ‘t Hooft

We postulate the existence of an ontological basis.
It is an orthonormal basis of Hilbert space that is truly superior to the basis choices that we are familiar with. In terms of an ontological basis, the evolution operator for a sufficiently fine mesh of time variables, does nothing more than permute the states.

p. 66, The Cellular Automaton Interpretation of Quantum Mechanics. Springer, 2016

To 3

Hermann Weyl

Our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group. . . .
Because of these results I feel certain that the general scheme of quantum kinematics formulated above is correct. But the field of discrete groups offers many possibilities which we have not as yet been able to realize in Nature; perhaps these holes will be filled by applications to nuclear physics.

p. 276, The Theory of Groups and Quantum Mechanics. 1928, transl. Dover 1950

To 5

Tom Banks

RUNHETC-2020-03

Finite Deformations of Quantum Mechanics

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Abstract

We investigate modifications of quantum mechanics (QM) that replace the unitary group in a finite dimensional Hilbert space with a finite group and determine the minimal sequence of subgroups necessary to approximate QM arbitrarily closely for general choices of Hamiltonian. This mathematical study reveals novel insights about 't Hooft's Ontological Quantum Mechanics, and the derivation of statistical mechanics from quantum mechanics. We show that Kornyak's proposal to understand QM as classical dynamics on a Hilbert space of one dimension higher than that describing the universe, supplemented by a choice of the value of a naturally conserved quantum operator in that classical evolution, can probably be a model of the world we observe.

▸ To 1

Ordinary view of finite QM

Quantum state spaces are continuous, but they have some intriguing realisations of discrete structures hidden inside. . . . The structures we are aiming at are known under strange acronyms such as 'MUB' and 'SIC'.

p. 313, Bengtsson I., Zyczkowski K. Geometry of Quantum States. Cambridge University Press, 2006

▸ To 6