Analytical geometry of the projective space $\mathbb{RP}^3$ in terms of Plucker coordinates and geometric algebra

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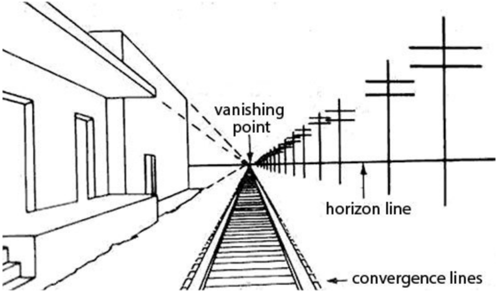
Outline of the report

- Motivation of work and methodological information.
- Classical analytic projective geometry (model $\mathbb{R}^3P$).
- Projective geometry in terms of geometric algebra.
- The software implementation for visualization in the Asymptote language.
- Results of work and prospects for further research.

Motivation of work

- We study the apparatus of geometric algebra in our scientific work.
- Variants of geometric algebra for Euclidean and pseudo-Euclidean spaces are well developed: there are educational materials, software libraries, including for CAS.
- On the contrary, the geometric algebra variant for projective geometry is poorly developed.
- Projective geometry – mathematical model of modern computer graphics.
- The authors are developing a course «Computer Geometry» which logically includes a section on projective geometry.



Perspective (latin *perspicere* – look through) – technique of depicting spatial objects in accordance with the distortion of the proportions and shape of the depicted bodies in their visual perception.

- It has been methodically applied in art and architecture since the Renaissance.
- Mathematical model of perspective – projective geometry.
- Computer graphics adopted the methods of fine art and projective geometry.

J. Desargues, J.-V. Poncelet, M. Chasles, K. G. H. von Staudt, A. F. Möbius, Y. Plücker, F. H. Klein contributed to the creation of the foundations of projective geometry.

Key advantages in terms of computing:

- eliminates exceptional cases (for example, all lines on the plane intersect);
- as a consequence, algorithms are simplified, often reduced to a single formula;
- affine transformations turn into linear ones;
- there is no division operation in formulas;
- as a consequence, it is possible to implement calculations for integer arithmetic.

Synthetic and analytical approaches to geometry

Synthetic approach

- Introduces a variety of geometric elements of various kinds: points, straight lines and planes.
- Sets the relationship between them in the form of axioms.
- Relying on axioms prove statements based on logic, geometric constructions and reasoning.

Analytical approach

- Is more correct — linear-algebraic (algebraic).
- Geometry problems are solved by algebraic methods («requires more ink, but less thinking»[a]).
- It is the most natural for algorithmization and implementation in software.
- The modern approach provides for a component-free record of formulas — simplification of programming, the possibility of vectorization (GPU, SIMD).

[a] Щербаков Р. Н., Пичурин Л. Ф. **От проективной геометрии — к неевклидовой (вокруг абсолюта).** / под ред. Т. А. Бурмистрова. Москва : Просвещение, 1979. 158 с. (Мир знаний), с. 89.

Two approaches can be distinguished:

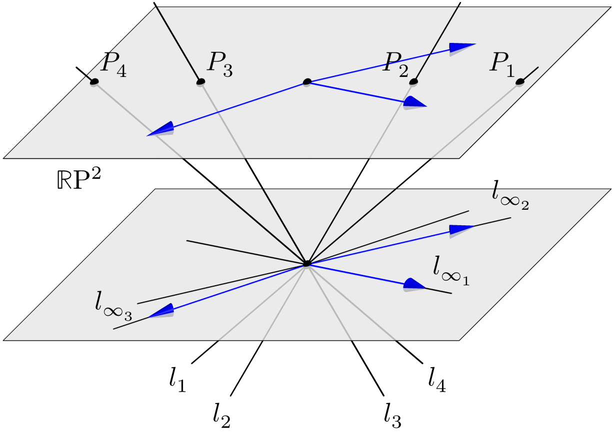
- Classical: based on homogeneous coordinates (Plucker coordinates, 6-vectors).
- Geometric algebra: based on the multivector algebra — special cases of Clifford algebra.

The second approach is poorly developed. It is handled by computer graphics specialists. Terminology and designations are not well established. Two techniques can be distinguished.

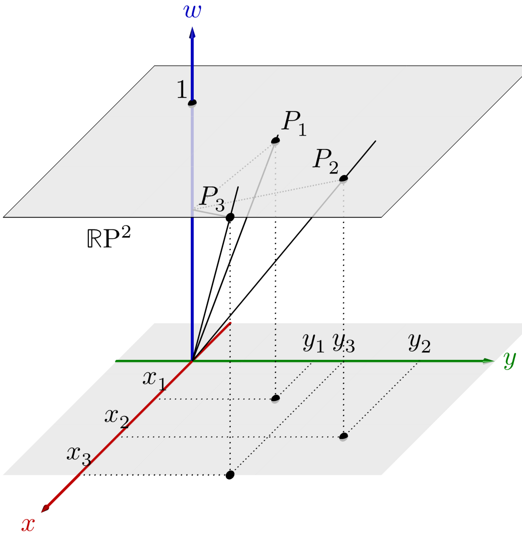
- Leo Dorst, Steven De Keninck и др., site `https://bivector.net/`.
- Eric Lengyel. Available developments in poster form link and posts in blog Projective Geometric Algebra Done Right. A consistent presentation of analytic projective geometry and exterior algebra in the manual[1].

[1] Lengyel E. Foundations of Game Engine Development. In 4 vols. Vol. 1. **Mathematics.** Lincoln, California : Terathon Software LLC, 2016. 195 p. ISBN 9780985811747. URL: `http://foundationsofgameenginedev.com`.

- Both approaches are computationally equivalent.
- The approach based on Plucker coordinates is more difficult to interpret. When proving formulas, there is a constant switching between the Euclidean space $\mathbb{R}^3$ and the projective space $\mathbb{R}\mathrm{P}^3$.
- The approach based on projective geometry is more logical and allows for a clear algebraic and geometric interpretation.



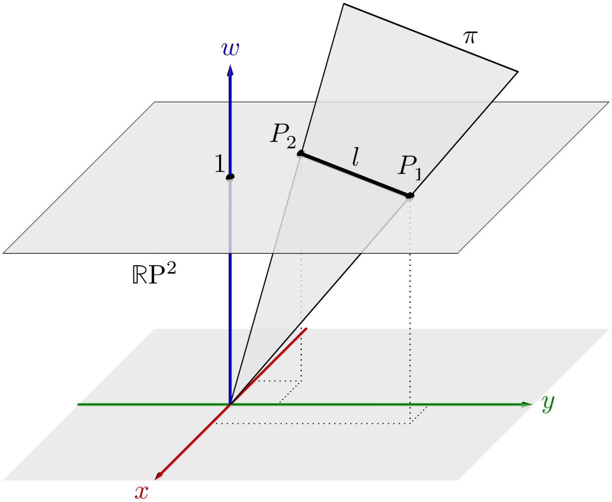
- The **model** of a projective **plane**.
- Figure illustrates a model of projective points.
- Finite points P_1, P_2, P_3, P_4.
- Infinity points (directions).
- All lines intersect at infinity.
- All infinity points lie on the infinity line (absolute).



The projective plane is modeled by a three-dimensional Cartesian space.

- Points vectors $\vec{\mathbf{p}} = (x, y, w)$.
- Directions vectors $\vec{\mathbf{v}} = (x, y, 0)$.
- Homogeneous coordinates $\vec{\mathbf{p}} = (x/w, y/w, 1) = (x : y : 1)$.
- It is easy to move from Cartesian space to projective space using normalization by division by weight w.

Line in $\mathbb{R}\mathrm{P}^2$



- Lines on $\mathbb{R}\mathrm{P}^2$ are planes $Ax + By + Cw = 0$ with $w = 1$ in the enclosing Cartesian space.
- Hence, the homogeneous equation of the line: $Ax + By + C = 0$.
- The guiding vector of the normal $\mathbf{N} = (A, B)$.
- Distance from O to the line: $d = C/\|N\|$.
- The normal equation of a straight line $n_x x + n_y y + d = 0$, where $\mathbf{n} = \dfrac{\mathbf{N}}{\|\mathbf{N}\|}$.

Plücker coordinates in $\mathbb{RP}^3$

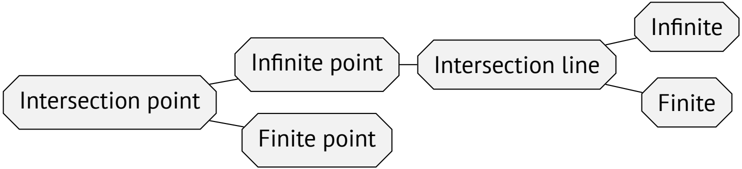
- It is necessary to use the model of the projective space $\mathbb{RP}^3$ and set the projective form of lines and planes.
- The line acquires a projective form in Plucker coordinates (another name is Grassmann coordinates).

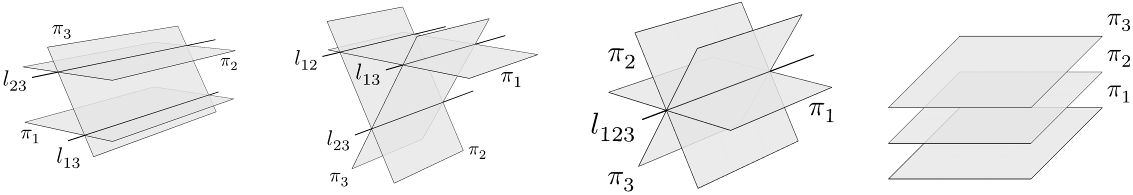
- Points: $\vec{\mathbf{p}} = (\mathbf{p} \mid w) = (x : y : z : w)$, origin point $\vec{\mathbf{O}} = (0 : 0 : 0 : 1)$.
- Line: $\vec{\mathbf{l}} = \{\mathbf{v} \mid \mathbf{m}\}$.
  - It takes 6 parameters for line: $\mathbf{v} = (v_x, v_y, v_z)^T$ and $\mathbf{m} = (m_x, m_y, m_z)^T$.
  - Plücker equation $(\mathbf{v}, \mathbf{m}) = 0$.
  - Vector $\mathbf{v}$ is direction vector, $\mathbf{m}$ is moment of line.
  - Tuple $(\mathbf{v}, \mathbf{m})$ is dual vector or motor from screw theory.
- The plane $\vec{\boldsymbol{\pi}} = [\mathbf{n} \mid d]$ is the normal equation of the plane, where $\mathbf{n}$ is the normal, and d is the distance from the plane to the origin.

The representation of a straight line as $\{\mathbf{v} \mid \mathbf{m}\}$ is not intuitive and does not have clear geometric interpretation.

		Формула	Описание
A	(2.1)	$\{\mathbf{v} \mid \mathbf{p} \times \mathbf{v}\}$	Прямая, проходящая через точку P по направлению $\mathbf{v}$.
B	(2.2)	$\{\mathbf{p}_2 - \mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Прямая, проходящая через две точки P_1 и P_2.
C	(2.6)	$\{\mathbf{p} \mid \mathbf{0}\}$	Прямая, проходящая через начало координат и точку P.
D	(2.3)	$\{w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Прямая, проходящая через две точки $\vec{\mathbf{p}}_1 = (\mathbf{p}_1 \mid w_1)$ и $\vec{\mathbf{p}}_2 = (\mathbf{p}_2 \mid w_2)$.
E	(2.15)	$\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$	Прямая линия пересечения двух плоскостей $[\mathbf{n}_1 \mid d_1]$ и $[\mathbf{n}_2 \mid d_2]$.
F	(2.16)	$(\mathbf{m} \times \mathbf{n} + d\mathbf{v} \mid -(\mathbf{n}, \mathbf{v}))$	Точка пересечения плоскости $[\mathbf{n} \mid d]$ и прямой $\{\mathbf{v} \mid \mathbf{m}\}$.
F.a	(2.24)	$[\mathbf{m}_1 \times \mathbf{m}_2 \mid (\mathbf{v}_2, \mathbf{m}_1)]$	Точка пересечения двух прямых $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ и $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$
F.b	(2.13)	$(d_1\mathbf{n}_3 \times \mathbf{n}_2 + d_2\mathbf{n}_1 \times \mathbf{n}_3 + d_3\mathbf{n}_2 \times \mathbf{n}_1 \mid (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3))$	Точка пересечения трех плоскостей $[\mathbf{n}_1 \mid d_1]$, $[\mathbf{n}_2 \mid d_2]$ и $[\mathbf{n}_3 \mid d_3]$
G	(2.4)	$(\mathbf{v} \times \mathbf{m} \mid (\mathbf{v}, \mathbf{v}))$	Точка, ближайшая к началу координат на прямой $\{\mathbf{v} \mid \mathbf{m}\}$.
H	(2.17)	$(-d\mathbf{n} \mid (\mathbf{n}, \mathbf{n}))$	Точка, ближайшая к началу координат на плоскости $[\mathbf{n} \mid d]$.
I	(2.19)	$[\mathbf{v} \times \mathbf{u} \mid -(\mathbf{u}, \mathbf{m})]$	Плоскость, содержащая прямую $\{\mathbf{v} \mid \mathbf{m}\}$ и направление $\mathbf{u}$.
J	(2.20)	$[\mathbf{v} \times \mathbf{p} + \mathbf{m} \mid -(\mathbf{p}, \mathbf{m})]$	Плоскость, содержащая прямую $\{\mathbf{v} \mid \mathbf{m}\}$ и точку $(\mathbf{p} \mid 1)$.
K	(2.21)	$[\mathbf{m} \mid 0]$	Плоскость, содержащая прямую $\{\mathbf{v} \mid \mathbf{m}\}$ и начало координат.
L	(2.18)	$[\mathbf{v} \times \mathbf{p} + w\mathbf{m} \mid -(\mathbf{p}, \mathbf{m})]$	Плоскость, содержащая прямую $\{\mathbf{v} \mid \mathbf{m}\}$ и точку $(\mathbf{p} \mid w)$.
L.a	(2.22)	$[\mathbf{v} \times \mathbf{u} \mid (\mathbf{u}, \mathbf{v}, \mathbf{p})]$	Плоскость, содержащая точку $(\mathbf{p} \mid 1)$ и направления $\mathbf{v}$ и $\mathbf{u}$.
M		$[\mathbf{m} \times \mathbf{v} \mid (\mathbf{m}, \mathbf{m})]$	Плоскость с прямой $\{\mathbf{v} \mid \mathbf{m}\}$, наиболее отдаленная от O.
N		$[-w\mathbf{p} \mid (\mathbf{p}, \mathbf{p})]$	Плоскость с точкой $(\mathbf{p} \mid w)$, наиболее отдаленная от O.
O	(2.23)	$\frac{\lvert(\mathbf{v}_1, \mathbf{m}_2) + (\mathbf{v}_2, \mathbf{m}_1)\rvert}{\lVert\mathbf{v}_1 \times \mathbf{v}_2\rVert}$	Расстояние между прямыми $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ и $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$.
P	(2.7)	$\frac{\lvert\mathbf{v} \times \mathbf{p} + \mathbf{m}\rvert}{\lVert\mathbf{v}\rVert}$	Расстояние между прямой $\{\mathbf{v} \mid \mathbf{m}\}$ и точкой $(\mathbf{p} \mid 1)$.
Q	(2.5)	$\lVert\mathbf{m}\rVert / \lVert\mathbf{v}\rVert$	Расстояние от прямой $\{\mathbf{v} \mid \mathbf{m}\}$ до начала координат.
R	(2.9)	$\frac{\lvert(\mathbf{n}, \mathbf{p}) + d\rvert}{\lVert\mathbf{n}\rVert}$	Расстояние от плоскости $[\mathbf{n} \mid d]$ до точки $(\mathbf{p} \mid 1)$.
S	(2.8)	$\lvert d\rvert / \lVert\mathbf{n}\rVert$	Расстояние от плоскости $[\mathbf{n} \mid d]$ до начала координат.

In a projective space, all lines intersect at a proper or non-proper point.





The duality principle in projective geometry

Для проективной **плоскости** двойственны термины:

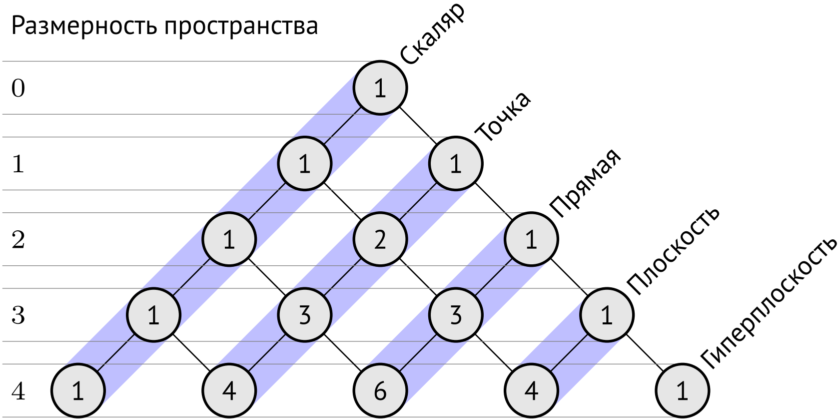
- «point» $\longleftrightarrow$ «line»;
- «point on the line» $\longleftrightarrow$ «line pass through point» $\longleftrightarrow$ «point and line is incident»;

For the projective **plane**, the terms are dual:

- ”dot” $\longleftrightarrow$ ”line”;
- ”the point lies on the line” $\longleftrightarrow$ ”the line passes through the point” $\longleftrightarrow$ ”the line and the point are incident”;

In the projective **space**, the duality principle can be formulated as follows:

Rank	**Original**	**Dual**
4	Point $(\mathbf{p} \mid w)$	Plane $[\mathbf{p} \mid w]$
6	Line $\{\mathbf{v} \mid \mathbf{m}\}$	Line $\{\mathbf{m} \mid \mathbf{v}\}$
4	Plane $[\mathbf{n} \mid d]$	Point $(\mathbf{n} \mid d)$



It is important to note $\dim \mathbb{R}\mathrm{P}^2 = 3$ and $\dim \mathbb{R}\mathrm{P}^3 = 4$.

Basic concepts.

- Exterior algebra (Grassmann algebra) is linear space with the operation of the exterior (wedge) product of $\wedge$ and $p-$vectors (or $p-$forms).
- Geometric algebra is linear space with geometric product operation and multivectors (graded algebra).
- For an orthonormal basis, it is sufficient to define a geometric product for vectors. Further, it constructively applies to any multivectors.

Особенности.

- GA is a specialized version of tensor algebra, easier to master.
- Generalization of complex numbers (elliptic, parabolic, hyperbolic), quaternions, biquaternions, screws, etc.
- GA can be used to describe Minkowski space, relativistic form of Maxwell's equations, and other fields of physics.

Wedge product

Wedge product of vectors from linear space L is operator $\wedge$, defined for every vectors $\mathbf{v}$, $\mathbf{u}$ и $\mathbf{w}$ from L whith fallowing properties:

1. $1 \wedge \mathbf{u} = \mathbf{u} \wedge 1 = \mathbf{u}$, where $1 \in \mathbb{R}$;
2. $\alpha \wedge \beta = \alpha \cdot \beta$, where $\alpha, \beta \in \mathbb{R}$;
3. $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$ – associativity;
4. $(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}$ – right distributivity;
5. $\mathbf{w} \wedge (\mathbf{v} + \mathbf{u}) = \mathbf{w} \wedge \mathbf{v} + \mathbf{w} \wedge \mathbf{u}$ – left distributivity;
6. $\mathbf{u} \wedge (\alpha \mathbf{v}) = (\alpha \mathbf{u}) \wedge \mathbf{v} = \alpha(\mathbf{u} \wedge \mathbf{v})$;
7. $\boxed{\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}}$ – anticommutativity (antisymmetry, skew symmetry).

The antisymmetry property is equivalent to the following:

$$\mathbf{u} \wedge \mathbf{u} = 0.$$

Проективное пространство в терминах геометрической алгебры

L is linear space $\dim L = 4$, basis vectors $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$. Scalar product operation, without the condition of positive definiteness. For the basis vectors, we assume:

$$(\mathbf{e}_1, \mathbf{e}_1) = (\mathbf{e}_2, \mathbf{e}_2) = (\mathbf{e}_3, \mathbf{e}_3) = 1, \quad (\mathbf{e}_4, \mathbf{e}_4) = 0, \text{ and } \mathbf{e}_4 \neq \mathbf{0}.$$

The outer product of the basis vectors

Type	Basis	Rank
Скаляр	1	0/4
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$	1/3
Bivectors	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$	2/2
Trivectors	$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$ $\mathbf{e}_{124} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$ $\mathbf{e}_{314} = \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4$ $\mathbf{e}_{234} = \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	3/1
Pseudoscalars	$\mathbf{e}_{1234} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4/0

Geometric product of basis vectors

Geometric product of orthonormal basis vectors

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_{13} = -\mathbf{e}_{31}$	$\mathbf{e}_{14} = -\mathbf{e}_{41}$
$\mathbf{e}_2$	$\mathbf{e}_{21}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_{24} = -\mathbf{e}_{42}$
$\mathbf{e}_3$	$\mathbf{e}_{31}$	$\mathbf{e}_{32} = -\mathbf{e}_{23}$	1	$\mathbf{e}_{34} = -\mathbf{e}_{43}$
$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0

Point

Defined by a vector (4 components)

$$\vec{\mathbf{p}} = x\mathbf{e}_1 + y\mathbf{e}_y + z\mathbf{e}_z + w\mathbf{e}_4,$$

where $\mathbf{p} = (x, y, z)$ is radius-vector $\mathbb{R}^3$ и w is width.

Line

Defined by a bivector (6 components)

$$\vec{\mathbf{l}} = \mathbf{v}_x\mathbf{e}_{41} + \mathbf{v}_y\mathbf{e}_{42} + \mathbf{v}_z\mathbf{e}_{43} + \mathbf{m}_x\mathbf{e}_{23} + \mathbf{v}_y\mathbf{e}_{31} + \mathbf{m}_z\mathbf{e}_{12},$$

where $\mathbf{v} = (v_x, v_y, v_z)$ is direction vector, $\mathbf{m} = (m_x, m_y, m_z)$ is line moment.

Plane

Defined by trivector (4 components)

$$\vec{\boldsymbol{\pi}} = \mathbf{n}_x\mathbf{e}_{234} + \mathbf{n}_y\mathbf{e}_{314} + \mathbf{n}_z\mathbf{e}_{321} + d\mathbf{e}_{321},$$

where $\mathbf{n}$ is unit normal vector, d is distance from coordinates origin to plane.

There are a number of packages for open CAS systems that implement the apparatus of geometric algebra.

- For Python and SymPy – GAlgebra
- For Julia – Grassmann.jl
- For Maxima – Clifford

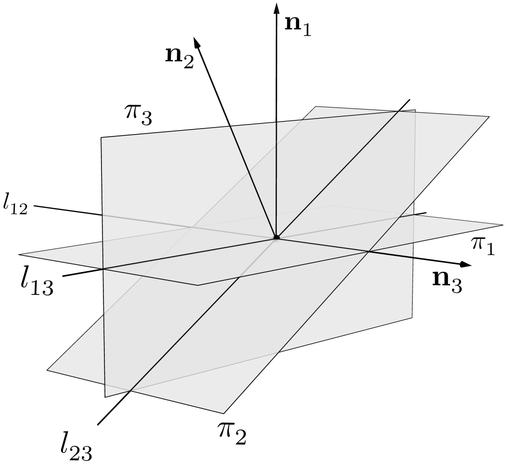
No system known to the authors supports the case of a projective space.

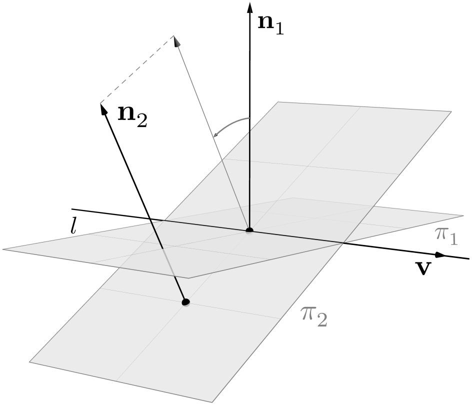
Libruary for Asymptote

The authors have implemented a small library for the Asymptote language (a language for creating vector two-dimensional and three-dimensional illustrations).

- The structures of `Line2D`, `Line3D` and `Plane 3D` are defined.
- Uses an approach based on classical analytic projective geometry.
- Defined functions linked to structures that implement formulas from the table above.
- Overloaded some standard Asymptote functions, which allows you to visualize the results of calculations.

Some examples of illustrations





Results

- Detailed derivation of analytical projective geometry formulas based on the material and notation from Foundations of Game Engine Development.
- As a result, the table from **Lengyel:Game Engine:v1:2016** is supplemented and expanded.
- A library has been written for the Asymptote language. Library allows to set points, lines and planes in homogeneous coordinates, find intersections and visualize the results.

Further work.

- Output of similar formulas in terms of geometric algebra.
- Connection with biquaternions and screw theory.
- Implementation of a prototype module for the SymPy computer algebra system (possibly based on GAlgebra).