The dual quaternion algebra and its implementation in Asymptote language

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Motivation of work

When studying geometric algebra (Clifford algebra), the authors constantly encountered mentions of screws and dual quaternions, but without any specific mathematics. This prompted us to find materials and consistently understand these concepts.

Goal

Implement the dual quaternion algebra programmatically and apply it to solving some problems on the movement of 3D objects in space

Tasks

- To consistently outline the theory of dual quaternions and match them with projective geometry, Euclidean geometry, geometric algebra.
- Implement the quaternion algebra programmatically.
- To apply them to free-body movement (screw motion).

Dual numbers

Consider a complex number of the following form $z = a + ib$, where $a, b \in \mathbb{R}$, i is a special number that is determined by the following property:

- if $i^2 = -1$, then we get the usual complex numbers (elliptical),
- if $i^2 = 1$ and at the same time $i \neq 1$ then we get hyperbolic (double, paracomplex, split) numbers,
- if $i^2 = 0$ and at the same time $i \neq 0$, then we get parabolic (dual) numbers.

Dual quaternion theory requires knowledge of the properties of dual numbers and quaternions.

Definition of a dual number

A dual number is a number of the following type

$$z = a + b\varepsilon,$$

where a and b are real numbers, and ε is a parabolic imaginary unit defined by the identity $\varepsilon^2 = 0$, with $\varepsilon \neq 0$.

- Number a is main or valid part.
- Number b is the dual or momentary part.
- The special number ε is also called the Clifford complexity symbol.

Sources: [1; 3—6]

Addition, subtraction and multiplication

Let be given two dual numbers $z_1 = a_1 + b_1\varepsilon$ and $z_2 = a_2 + b_2\varepsilon$.

- addition: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\varepsilon$.
- subtraction: $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)\varepsilon$.
- multiplication: $z_1 z_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2)\varepsilon$.

The latter is true by virtue of

$$z_1 z_2 = (a_1 + b_1\varepsilon)(a_2 + b_2\varepsilon) = a_1 a_2 + a_1 b_2 \varepsilon + b_1 a_2 \varepsilon + b_1 b_2 \varepsilon^2 = a_1 a_2 + (a_1 b_2 + b_1 a_2)\varepsilon,$$

since $\varepsilon^2 = 0$.

Using the multiplication formula, one can calculate the square of a dual number:

$$z^2 = zz = a^2 + 2ab\varepsilon.$$

Conjugation, modulus, and inverse multiplication

- Dual conjugation or just conjugation is the following unary operation

$$\bar{z} = \overline{a + b\varepsilon} = a - b\varepsilon.$$

- Modulo a dual number is called the real number $|z| = \sqrt{z\bar{z}} = |a|$.
- is also often more convenient to use an expression for the square of a module:

$$|z|^2 = z\bar{z} = (a + b\varepsilon)(a - b\varepsilon) = a^2 - ab\varepsilon + ab\varepsilon - b^2\varepsilon^2 = a^2.$$

- Inverse is calculated by multiplying the dual number z^{-1}

$$z^{-1} = \frac{1}{a} - \frac{b}{a^2}\varepsilon.$$

The number z^{-1} is from the following relation:

$$1 = \frac{z\bar{z}}{z\bar{z}} = z\frac{\bar{z}}{z\bar{z}} = z\frac{\bar{z}}{|z|^2} \Rightarrow z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a - b\varepsilon}{a^2} = \frac{1}{a} - \frac{b}{a^2}\varepsilon.$$

Note also that you can calculate the real part by finding the difference $z + \bar{z}$:

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + B\varepsilon + a - b\varepsilon) = a.$$

Since the inverse of the multiplication of a dual number is calculated as

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

then the prerequisite for its existence is the condition $|z| \neq 0$.
The division operation is defined by the following relation:

$$\frac{z_1}{z_2} = \frac{a_1 + b_1\varepsilon}{a_2 + b_2\varepsilon} = \frac{a_1}{a_2} + \frac{b_1 a_2 - b_2 a_1}{a_2^2}\varepsilon,$$

which is true due to the following chain of equalities:

$$\frac{a_1 + b_1\varepsilon}{a_2 + b_2\varepsilon} = \frac{(a_1 + b_1\varepsilon)(a_2 - b_2\varepsilon)}{(a_2 + b_2\varepsilon)(a_2 - b_2\varepsilon)} = \frac{a_1 a_2 - a_1 b_2\varepsilon + b_1 a_2\varepsilon}{a_2^2} = \frac{a_1}{a_2} + \frac{b_1 a_2 - b_2 a_1}{a_2^2}\varepsilon.$$

An analogue of the trigonometric or exponential form of a number

For a dual number z with a nonzero module $|z| \neq 0$, we can write:

$$z = a + b\varepsilon = a\left(1 + \frac{b}{a}\varepsilon\right) = r(1 + \varphi\varepsilon), \quad r = |z| = |a|, \quad \varphi = \frac{b}{a}.$$

The number $\varphi = \operatorname{Arg} z = \dfrac{b}{a}$ – argument of a dual number (or parameter of a number).

- Conjugation: $\bar{z} = \overline{r(1 + \varphi\varepsilon)} = r(1 - \varphi\varepsilon)$.
- Multiplication: $r_1(1 + \varphi_1\varepsilon)r_2(1 + \varphi_2\varepsilon) = r_1r_2(1 + (\varphi_1 + \varphi_2)\varepsilon)$
- Division: $\dfrac{z_1}{z_2} = \dfrac{r_1(1 + \varphi_1\varepsilon)}{r_2(1 + \varphi_2\varepsilon)} = \dfrac{r_1}{r_2}(1 + (\varphi_1 - \varphi_2)\varepsilon)$.

The division expression is valid by virtue of the following chain of equalities:

$$\frac{z_1}{z_2} = \frac{r_1(1 + \varphi_1\varepsilon)}{r_2(1 + \varphi_2\varepsilon)} = \frac{r_1(1 + \varphi_1\varepsilon)r_2(1 - \varphi_2\varepsilon)}{r_2r_2(1 + \varphi_2\varepsilon)(1 - \varphi_2\varepsilon)} = \frac{r_1(1 + (\varphi_1 - \varphi_2)\varepsilon)}{r_2(1 - \varphi_2\varepsilon + \varphi_2\varepsilon)} = \frac{r_1}{r_2}(1 + (\varphi_1 - \varphi_2)\varepsilon)$$

Exponentiation

Since in exponential form, multiplication of a dual number is reduced to adding arguments and multiplying modules, then for exponentiation we can write:

$$\big(r(1+\varphi\varepsilon)\big)^n = \underbrace{r\cdot r\cdot\ldots\cdot r}_{n}\big(1+(\varphi+\varphi+\cdots+\varphi)\varepsilon\big) = r^n(1+n\varphi\varepsilon).$$

Hence:

$$z^n = \big(r(1+\varphi\varepsilon)\big)^n = r^n\big(1+n\varphi\varepsilon\big)$$

$$z^n = (a+b\varepsilon)^n = a^n + na^{n-1}b\varepsilon.$$

In particular:

$$(a+b\varepsilon)^2 = a^2 + 2ab\varepsilon.$$

For negative degrees:

$$(a+b\varepsilon)^{-n} = \left(\frac{1}{a}-\frac{b}{a^2}\varepsilon\right)^n = \frac{1}{a^n} - n\frac{1}{a^{n-1}}\frac{b}{a^2}\varepsilon = \frac{1}{a^n} - n\frac{b}{a^{n+1}}\varepsilon,$$

in particular:

$$(a+b\varepsilon)^{-1} = \frac{1}{a}-\frac{b}{a^2}\varepsilon.$$

Root extraction

$$\sqrt[n]{r(1+\varphi\varepsilon)} = \sqrt[n]{r}\left(1+\frac{\varphi}{n}\varepsilon\right),$$

$$\sqrt[n]{a+\varepsilon b} = \sqrt[n]{a}\left(1+\frac{b\varepsilon}{na}\right) = \sqrt[n]{a}+\frac{b}{n}a^{\frac{1-n}{n}}\varepsilon,$$

$$\sqrt{a+\varepsilon b} = \sqrt{a}\left(1+\frac{b\varepsilon}{2a}\right) = \sqrt{a}+\frac{b}{2\sqrt{a}}\varepsilon,$$

$$(a+b\varepsilon)^{\frac{n}{m}} = a^{\frac{n}{m}}\left(1+\frac{n}{m}\frac{b}{a}\varepsilon\right) = a^{\frac{n}{m}}+\frac{n}{m}ba^{\frac{n}{m}-1}\varepsilon.$$

Elementary functions of a dual number

The formula $f(a+\varepsilon b) = f(a) + f'(a)b\varepsilon$ allows you to extend elementary functions to a set of dual numbers, since on the right side of the formula there are only values of the function f from the real number a. To illustrate, here is a small summary of the basic elementary functions.

Trigonometric functions	**Inverse trigonometric functions**
$\sin(a+b\varepsilon) = \sin a + b\cos a\varepsilon$	$\arcsin(a+b\varepsilon) = \arcsin a + b\varepsilon/\sqrt{1-a^2}$
$\cos(a+b\varepsilon) = \cos a - b\varepsilon\sin a$	$\arccos(a+b\varepsilon) = \arccos a - b\varepsilon/\sqrt{1-a^2}$
$\operatorname{tg}(a+b\varepsilon) = \operatorname{tg} a + b\varepsilon/\cos^2 a$	$\operatorname{arctg}(a+b\varepsilon) = \operatorname{arctg} a + b\varepsilon/(1+a^2)$
$\operatorname{ctg}(a+b\varepsilon) = \operatorname{ctg} a - b\varepsilon/\sin^2 a$	$\operatorname{arcctg}(a+b\varepsilon) = \operatorname{arctg} a - b\varepsilon/(1+a^2)$

Power functions	**Logarithmic functions and exponent**
$(a+b\varepsilon)^n = a^n + na^{n-1}b\varepsilon$	$\exp(a+b\varepsilon) = \exp\{a\} + b\exp\{a\}\varepsilon$
$\sqrt[n]{a+b\varepsilon} = \sqrt[n]{a}\left(1+\frac{b\varepsilon}{na}\right)$	$\log_c(a+\varepsilon b) = \log_c a + b\varepsilon/a\ln a$

Quaternions

Quaternions

A quaternion (elliptical quaternions) is a hypercomplex number of the following form

$$q = q_0 1 + q_1 \mathrm{i} + q_2 \mathrm{j} + q_3 \mathrm{k} = q_0 + \mathbf{q},$$

where q_0, q_1, q_2, q_3 are some real numbers.

The quaternion q can also be associated with a point in projective space written in homogeneous coordinates $(q_1, q_2, q_3 \mid q_0)$. In this case, the base element 1 is associated with its own (final) origin point, and the base elements i, j, k with points at infinity:

$$1 \leftrightarrow O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathrm{i} \leftrightarrow \vec{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathrm{j} \leftrightarrow \vec{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathrm{k} \leftrightarrow \vec{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Quaternion multiplication

The rules of multiplication of basic elements $\langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k} \rangle$ are derived from Hamilton's axiomatic relation:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1.$$

The multiplication table of quaternionic basic elements will take the form:

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

The formula for quaternion multiplication:

$$pq = (p_0 q_0 - (\mathbf{p}, \mathbf{q})) + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}.$$

In particular, for pure quaternions $p = 0 + \mathbf{p}$ and $q = 0 + \mathbf{q}$, the formula is simplified:

$$\mathbf{pq} = -(\mathbf{p}, \mathbf{q}) + \mathbf{p} \times \mathbf{q}.$$

Conjugation, square and module

Let's introduce the operation of quaternionconjugation. If the quaternion $p = p_0 + \mathbf{p}$ is given, then its conjugation is determined by the following formula:

$$p^* = p_0 - \mathbf{p} = p_0 - p_1 \mathrm{i} - p_2 \mathrm{j} - p_3 \mathrm{k}.$$

The module of a quaternion is the expression

$$|p| = \sqrt{pp^*} = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2},$$

and the norm of a pure quaternion is the expression

$$\|\mathbf{p}\| = \sqrt{(\mathbf{p}, \mathbf{p})} = \sqrt{p_1^2 + p_2^2 + p_3^2}.$$

We define the scalar product of quaternions by the following formula

$$(p, q) = \frac{1}{2}(pq^* + qp^*) = p_0 q_0 + (\mathbf{p}, \mathbf{q}) = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3.$$

Rotation using quaternions

A unit quaternion of the following form

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a}, \quad \mathbf{a} = a_x\mathrm{i} + a_y\mathrm{j} + a_z\mathrm{k},$$

sets the rotation of the point P using the sandwich operator

$$p' = qpq^*,$$

where the quaternion $p = 1 + \mathbf{p} = 1 + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ is associated with an affine point P, θ is the angle of rotation around an axis passing through the origin and having a guiding vector $\mathbf{a} = (a_x, a_y, a_z)$, where $\|\mathbf{a}\| = 1$.

Dual quaternions

Parabolic (dual) biquaternions

Consider a dual number with coefficients in the form of quaternions:

$$Q = q + q^{o}\varepsilon, \quad q, q^{o} \in \mathbb{H}, \quad \varepsilon^{2} = 0, \quad \varepsilon \neq 0,$$

where $q = q_0 + \mathbf{q} = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ – main part и $q^{o} = q_0^{o} + \mathbf{q}^{o} = q_0^{o} + q_1^{o}\mathrm{i} + q_2^{o}\mathrm{j} + q_3^{o}\mathrm{k}$ – the moment part. Q It can be written as a number with 8 components:

$$Q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} + q_0^{o}\varepsilon + q_1^{o}\mathrm{i}\varepsilon + q_2^{o}\mathrm{j}\varepsilon + q_3^{o}\mathrm{k}\varepsilon.$$

The hypercomplex number Q is called parabolic biquaternion, as well as dual quaternion and dual biquaternion. One can also consider the basic elements $\langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k}, \varepsilon, \mathrm{i}\varepsilon, \mathrm{j}\varepsilon, \mathrm{k}\varepsilon \rangle$ and make a complete multiplication table of 8×8 elements.

	1	i	j	k	ε	$\mathrm{i}\varepsilon$	$\mathrm{j}\varepsilon$	$\mathrm{k}\varepsilon$
1	1	i	j	k	ε	$\mathrm{i}\varepsilon$	$\mathrm{j}\varepsilon$	$\mathrm{k}\varepsilon$
i	i	-1	k	$-\mathrm{j}$	$\mathrm{i}\varepsilon$	$-\varepsilon$	$\mathrm{k}\varepsilon$	$-\mathrm{j}\varepsilon$
j	j	$-\mathrm{k}$	-1	i	$\mathrm{j}\varepsilon$	$-\mathrm{k}\varepsilon$	$-\varepsilon$	$\mathrm{i}\varepsilon$
k	k	j	$-\mathrm{i}$	-1	$\mathrm{k}\varepsilon$	$\mathrm{j}\varepsilon$	$-\mathrm{i}\varepsilon$	$-\varepsilon$
ε	ε	$\mathrm{i}\varepsilon$	$\mathrm{j}\varepsilon$	$\mathrm{k}\varepsilon$	0	0	0	0
$\mathrm{i}\varepsilon$	$\mathrm{i}\varepsilon$	$-\varepsilon$	$\mathrm{k}\varepsilon$	$-\mathrm{j}\varepsilon$	0	0	0	0
$\mathrm{j}\varepsilon$	$\mathrm{j}\varepsilon$	$-\mathrm{k}\varepsilon$	$-\varepsilon$	$\mathrm{i}\varepsilon$	0	0	0	0
$\mathrm{k}\varepsilon$	$\mathrm{k}\varepsilon$	$\mathrm{j}\varepsilon$	$-\mathrm{i}\varepsilon$	$-\varepsilon$	0	0	0	0

The table is compiled using two assumptions:

- associativity of multiplication,
- the commutativity of multiplying a dual imaginary unit ε by elliptical imaginary units i, j, k that is:

$$\mathrm{i}\varepsilon = \varepsilon\mathrm{i}, \quad \mathrm{j}\varepsilon = \varepsilon\mathrm{j}, \quad \mathrm{k}\varepsilon = \varepsilon\mathrm{k}.$$

Conjugations and the scalar product of a biquaternion

In the case of biquaternions, three different conjugation operations are considered:

- $Q^* = (q + q^o\varepsilon) = q^* + q^{o*}\varepsilon$ – quaternionic (complex) conjugation;
- $\overline{Q} = \overline{q + q^o\varepsilon} = q - q^o\varepsilon$ – dual conjugation;
- $Q^\dagger = (\overline{q + q^o\varepsilon})^* = q^* - q^{o*}\varepsilon$ – quaternionic dual conjugation.

The following properties are valid for these operations:

$$(PQ)^* = Q^*P^*, \quad \overline{QP} = \overline{PQ}, \quad (PQ)^\dagger = Q^\dagger P^\dagger.$$

Scalar product of two quaternions $P = p + p^o\varepsilon$ and $Q = q + q^o\varepsilon$ is defined as follows [1, p. 15]:

$$(P, Q) = \frac{1}{2}(PQ^* + QP^*) = (p, q) + [(p^o, q) + (p, q^o)]\varepsilon,$$

where $(p, q) = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3$, $(p^o, q) = p_0^oq_0 + p_1^oq_1 + p_2^oq_2 + p_3^oq_3$ и $(p, q^o) = p_0q_0^o + p_1q_1^o + p_2q_2^o + p_3q_3^o$ – scalar products of quaternions [1, p. 15].

Biquaternion product of biquaternions

For two biquaternions $P = p + p^{o}\varepsilon$ и $Q = q + q^{o}\varepsilon$ one can define biquaternion product

$$PQ = (p + p^{o}\varepsilon)(q + q^{o}\varepsilon) = pq + (pq^{o} + p^{o}q)\varepsilon,$$

where pq, pq^{o} and $p^{o}q$ – quaternion products.

Module of biquaternion

The square of the biquaternion module is defined by the following expression:

$$|Q|^2 = QQ^* = qq^* + (qq^{o*} + q^o q^*)\varepsilon = |q|^2 + 2(q, q^o)\varepsilon,$$

where $|Q|^2$ – dual number. Directly biquaternion module will be calculated as follows:

$$|Q| = \sqrt{QQ^*} = \sqrt{|q|^2 + 2(q, q^o)\varepsilon} = |q| + \frac{(q, q^o)}{|q|}\varepsilon = |q|\left(1 + \frac{(q, q^o)}{|q|^2}\varepsilon\right),$$

which is true by virtue of the formula for dual numbers $\sqrt{a + b\varepsilon} = \sqrt{a}\left(1 + \frac{b}{2a}\varepsilon\right)$, где $a, b \in \mathbb{R}$.

The real number $\frac{(q,q^o)}{|q|^2}$ is called the biquaternion parameter, and the dot product of the quaternions is (q, q^o) is called the biquaternion invariant [5, c. 71].
If the biquaternion is pure, that is, $q_0 = 0$ and $q_0^o = 0$, then it is a screw $\mathbf{Q} = \mathbf{q} + \mathbf{q}^o\varepsilon$ and its parameter is the same as the screw parameter $\frac{(\mathbf{q}, \mathbf{q}^o)}{\|\mathbf{q}\|^2}$.
A biquaternion is called unit if its modulus is 1, that is $(q, q^o) = 0$ и $|q| = 1$.

In another way, dual biquaternions are obtained from quaternions $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ using the doubling procedure when replacing the real coefficients q_0, q_1, q_2, q_3 with dual numbers Q_0, Q_1, Q_2, Q_3.

$$Q = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3 k = Q_0 + \mathbf{Q}, \quad Q_i = q_i + q_i^o\varepsilon, \quad q_i, q_i^o \in \mathbb{R}, \quad i = 0, 1, 2, 3,$$

where Q_0 is the scalar part (dual number), and $\mathbf{Q}$ is the screw part (dual vector).
For two biquaternions

$$Q = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3 k = Q_0 + \mathbf{Q}, \quad \text{и} \quad P = P_0 + P_1\mathrm{i} + P_2\mathrm{j} + P_3 k = P_0 + \mathbf{P},$$

similarly to quaternions, we can prove the formula for biquaternion product

$$PQ = P_0Q_0 - (\mathbf{P}, \mathbf{Q}) + P_0\mathbf{Q} + Q_0\mathbf{P} + \mathbf{P} \times \mathbf{Q},$$

where $(\mathbf{P}, \mathbf{Q})$ is scalar, and $\mathbf{P} \times \mathbf{Q}$ is the vector product of the screws $\mathbf{P}$ and $\mathbf{Q}$. For pure biquaternions:

$$\mathbf{PQ} = -(\mathbf{P}, \mathbf{Q}) + \mathbf{P} \times \mathbf{Q}, \quad \mathbf{PP} = -\|\mathbf{P}\|^2.$$

The scalar product is calculated as follows:

$$(P, Q) = P_0Q_0 + P_1Q_1 + P_2Q_2 + P_3Q_3.$$

The product of dual numbers and biquaternions

Consider a biquaternion in the quaternion representation $Q = q + q^o\varepsilon$ and multiply it by the dual number $\mathrm{A} = \alpha + \alpha^o\varepsilon$:

$$(\alpha + \alpha^o\varepsilon)(q + q^o\varepsilon) = \alpha q + (\alpha q^o + \alpha^o q)\varepsilon.$$

Similarly in the dual representation $Q = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3\mathrm{k}$, $Q_i = q_i + q_i^o\varepsilon$, $i = 0, 1, 2, 3$ one can write:

$$\mathrm{A}Q_i = (\alpha + \alpha^o\varepsilon)(q_i + q_i^o\varepsilon) = \alpha q_i + (\alpha q_i^o + \alpha^o q_i)\varepsilon.$$

Quaternionic and dual representation of biquaternions

$$Q = q + q^o\varepsilon, \quad q, q^o \in \mathbb{H}, \quad \varepsilon^2 = 0, \quad \varepsilon \neq 0,$$

where q is main part, q^o is moment part.

$$q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}, \quad q^o = q_0^o + q_1^o\mathrm{i} + q_2^o\mathrm{j} + q_3^o\mathrm{k}.$$

Biquaternion multiplication:

$$PQ = pq + (pq^o + p^o q)\varepsilon,$$

where pq, pq^o, $p^o q$ is quaternion multiplication.
The scalar product of biquaternions:

$$(P, Q) = (p, q) + [(p^o, q) + (p, q^o)]\varepsilon,$$

(p, q), (p^o, q), (p, q^o) – scalar products of quaternions.
The square of the biquaternion module:

$$|Q|^2 = |q|^2 + 2(q, q^o)\varepsilon.$$

$$Q = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3\mathrm{k} = Q_0 + \mathbf{Q}$$

Q_0, Q_1, Q_2, Q_3 are dual numbers, Q_0 is scalar part, $\mathbf{Q}$ is screw part.
Biquaternions multiplication:

$$PQ = P_0Q_0 - (\mathbf{P}, \mathbf{Q}) + P_0\mathbf{Q} + Q_0\mathbf{P} + \mathbf{P} \times \mathbf{Q},$$

where $(\mathbf{P}, \mathbf{Q})$, $\mathbf{P} \times \mathbf{Q}$ are scalar and screw multiplication of screws, $P_0\mathbf{Q}$, $Q_0\mathbf{P}$ is multiplication of screw by a dual number.
The scalar product of biquaternions:

$$(P, Q) = P_0Q_0 + P_1Q_1 + P_2Q_2 + P_3Q_3$$

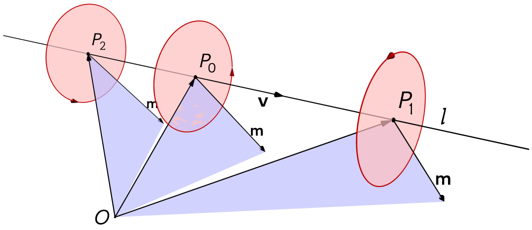
The square of the biquaternion module:

$$|Q|^2 = Q_0^2 + (\mathbf{Q}, \mathbf{Q}) = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$$

Biquaternion representation of a point, a straight line, and a plane

Biquaternions allow you to map different algebraic objects to a point, vector, straight line, and plane in the way shown in the table.

Geometric object	Biquaternion representation	Homogenies coordinates	Three-dimensional cartesian space
Affine point	$P = 1 + \mathbf{p}\varepsilon,\ \mathbf{p} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$	$\vec{\mathbf{p}} = (\mathbf{p} \mid 1) = (x, y, z \mid 1)$	$\mathbf{p} = (x, y, z)^T$
Mass-point	$P = w + \mathbf{p}\varepsilon$	$\vec{\mathbf{p}} = (\mathbf{p} \mid w) = (x, y, z \mid w)$	$\mathbf{p} = (x/w, y/w, z/w)$
Vector	$\mathbf{V} = \mathbf{v}\varepsilon,\ \mathbf{v} = v_x\mathrm{i} + v_y\mathrm{j} + v_z\mathrm{k}$	$\vec{\mathbf{v}} = (\mathbf{v} \mid 0) = (v_x, v_y, v_z \mid 0)$	$\mathbf{v} = (v_x, v_y, v_z)^T$
Line	$\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$ $P(t) = P_0 + \mathbf{v}t\varepsilon,$ $P_0 = 1 + \dfrac{\mathbf{v} \times \mathbf{m}}{\lVert\mathbf{v}\rVert^2}\varepsilon$	$\vec{\mathbf{L}} = \{\mathbf{v} \mid \mathbf{m}\}$ $\vec{\mathbf{p}} = (\mathbf{v} \times \mathbf{m} \mid \lVert\mathbf{v}\rVert^2)$	$\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{v}t$
Plane	$\Pi = \mathbf{n} + d\varepsilon,$ $\mathbf{n} = n_x\mathrm{i} + n_y\mathrm{j} + n_z\mathrm{k}$	$\vec{\boldsymbol{\pi}} = [\mathbf{n} \mid d]$	$ax + by + cz + d = 0$



For any pair of vectors $\{\mathbf{v} \mid \mathbf{m}\}$ for which the Plucker condition$(\mathbf{v}, \mathbf{m}) = 0$ is satisfied, a pure biquaternion can be mapped

$$\mathbf{L} = \mathbf{v} + \varepsilon \mathbf{m}$$

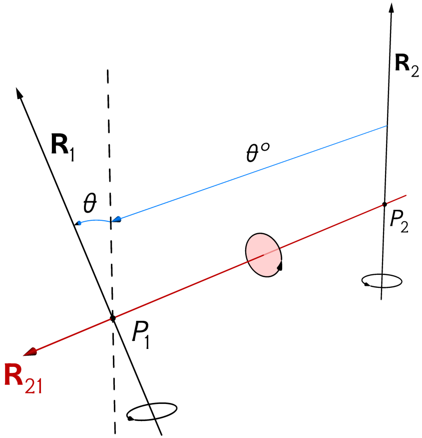
and unambiguously interpret it as an axis passing through a point with a radius vector $\mathbf{p} = \mathbf{v} \times \mathbf{m}$ in the direction of $\mathbf{v}$.

Six components

$$\{v_x, v_y, v_z \mod m_x, m_y, m_z\}$$

they are called Plucker coordinates.

The dual angle



The dual angle $\Theta = \theta + \theta^o\varepsilon$ between two axes $\mathbf{A}_1$ and $\mathbf{A}_2$ is a shape formed by these axes and a straight line segment P_2P_1 intersecting the axes at right angles.

- Pure biquaternion $\mathbf{A}_{21}$ with an axis in the form of a straight line (P_2P_1) – dual angle axis.
- The dual part of the angle $\theta^o = \|\mathbf{P_2P_1}\|$.
- The real part of the angle is $\theta = \measuredangle(\mathbf{A}_2, \mathbf{A}_1)$.

The dual angle Θ is defined by a pure biquaternion

$$\boldsymbol{\Theta} = \Theta\mathbf{A}_{21} = (\theta + \theta^o\varepsilon)\mathbf{A}_{21}.$$

The following formulas are used to calculate the trigonometric functions of the dual angle:

$$\sin\Theta = \sin(\theta + \theta^o\varepsilon) = \sin\theta + \theta^o\cos\theta\varepsilon,$$
$$\cos\Theta = \cos(\theta + \theta^o\varepsilon) = \cos\theta - \theta^o\sin\theta\varepsilon,$$
$$\operatorname{tg}\Theta = \operatorname{tg}(\theta + \theta^o\varepsilon) = \operatorname{tg}\theta + \frac{\theta^o}{\cos^2\theta}\varepsilon.$$

Transference principle

Some definitions from the theory of rigid body

Immutable system is a system of material points in which the distance between any two points is constant. With a continuous distribution of masses, such a system is an ideal image of a solid body and is called an emphabsolutely solid body [2, c. 48].
Rigid bodies are distinguished:

- with one fixed point,
- is free.

Euler's theorem

any movement of an absolutely rigid body near a fixed point can be obtained only by rotating the body around a certain axis passing through this point and called the axis of final rotation. [2, c. 132].

Chasles theorem

any movement of a free absolutely rigid body can be carried out by a single screw movement around a certain screw axis, called the axis of the final screw movement. [2, c. 153].

Kotelnikov–Study Transference principle

Transference principle

All formulas of the theory of finite rotations and kinematics of motion of a rigid body with one fixed point, when replacing real quantities with dual analogues of them, pass into formulas of the theory of finite displacements and kinematics of motion of a free rigid body. [5, c. 67].

In other words, if one replace real numbers, vectors, angles, and quaternions with dual numbers, pure biquaternions (screws), dual angles, and biquaternions in formulas for rotating a point in space, one will get correct formulas for screw motion.
The principle was formulated by Kotelnikov Alexander Petrovich and by Eduard Study [3, c. 12—13]. We have not found an explicit formulation of this principle in the English-language literature.

Dual quaternions screw motion

Let's apply the Kotelnikov-Study transfer principle to derive a formula for biquaternionic screw motion. It is known that the unit quaternion is of the following form

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a}, \quad \mathbf{a} = a_x\mathrm{i} + a_y\mathrm{j} + a_z\mathrm{k},$$

where θ is the angle of rotation around an axis passing through the origin and having a direction vector $\mathbf{a} = (a_x, a_y, a_z)^T$, where $\|\mathbf{a}\| = 1$, sets the rotation of the point P using the sandwich operator

$$p' = qpq^*,$$

where the quaternion $p = 1 + \mathbf{p} = 1 + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ defines the point P.
According to the principle of transfer, the biquaternion defining the screw motion (translation + rotation) will be obtained from the rotational quaternion by the following substitution:

- $\theta \longrightarrow \Theta = \theta + \theta^{O}\varepsilon$ – the angle is replaced by a dual angle;
- $\mathbf{a} \longrightarrow \mathbf{A} = \mathbf{a} + \mathbf{a}^{O}\varepsilon$ – the vector is replaced by a pure biquaternion (screw).

Biquaternion of screw motion (rotation + translation)

The biquaternion of screw motion is written as follows:

$$\Lambda = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{A}, \quad \Theta = \theta + \theta^{o}\varepsilon, \quad \mathbf{A} = \mathbf{a} + \mathbf{a}^{o}\varepsilon.$$

Since $\mathbf{A}$ defines a line, the Plucker condition $(\mathbf{a}, \mathbf{a}^{o}) = 0$ must be fulfilled, and we will also assume that $||\mathbf{a}|| = 1$, that is, $\mathbf{A}$ is a unit screw (a unit pure biquaternion).

- The dual number $\Theta = \theta + \theta^{o}\varepsilon$ is the dual angle with the axis $\mathbf{A}$.
- The angle θ sets the angle of rotation around the axis $\mathbf{A}$.
- The number θ^{o} sets the translation distance along the $\mathbf{A}$ axis.

Substitute expressions for $\sin\frac{\Theta}{2}$ and $\cos\frac{\Theta}{2}$ and write the biquaternion Λ in the following form:

$$\boxed{\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{o}\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^{o}}{2}\varepsilon}$$

- At $\theta^{o} = 0$ we get a pure rotation around an arbitrary axis $\mathbf{A}$ set by the quaternion $R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{o}\varepsilon)$.
- At $\theta = 0$ we get a pure translation along the $\mathbf{A}$ axis set by the biquaternion $T = 1 + \frac{\theta^{o}}{2}\mathbf{a}\varepsilon$.

Pure rotation of a point around an arbitrary axis

The affine point P is represented by the following biquaternion:

$$P = p + p^{o}\varepsilon = 1 + \mathbf{p}\varepsilon, \quad p = 1 + \mathbf{0}, \quad p^{o} = 0 + \mathbf{p}.$$

Rotation of a point around the axis $\mathbf{A} = \mathbf{a} + \mathbf{a}^{o}\varepsilon$ for the ordinary angle θ is carried out using the following sandwich formula:

$$P' = RPR^{\dagger}, \quad R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{o}\varepsilon), \quad R^{\dagger} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^{o}\varepsilon).$$

For a point:

$$P' = RPR^{\dagger} = 1 + \big(\cos\theta\mathbf{p} + \sin\theta\mathbf{a}\times\mathbf{p} + (1-\cos\theta)(\mathbf{a},\mathbf{p})\mathbf{a} + \sin\theta\mathbf{a}^{o} + (1-\cos\theta)\mathbf{a}\times\mathbf{a}^{o}\big)\varepsilon.$$

For the vector:

$$\mathbf{V}' = R\mathbf{V}R^{\dagger} = R\mathbf{v}\varepsilon R^{\dagger} = \big(\cos\theta\mathbf{v} + \sin\theta\mathbf{a}\times\mathbf{v} + (1-\cos\theta)(\mathbf{a},\mathbf{v})\mathbf{a}\big)\varepsilon,$$

Pure point and vector translation along the axis

Now consider the biquaternion

$$T = 1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon,$$

which sets the translation to a distance of θ^0 in the direction of $\mathbf{a}$. Here θ^o is the dual part of the dual angle Θ.
Point translation:

$$P' = TPT^{\dagger} = \left(1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon\right)(1 + \mathbf{p}\varepsilon)\left(1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon\right) = 1 + (\mathbf{p} + \theta^o\mathbf{a})\varepsilon.$$

Vector translation:

$$\mathbf{V}' = T\mathbf{V}T^{\dagger} = \left(1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon\right)\mathbf{v}\varepsilon\left(1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon\right) = \left(1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon\right)\left(\mathbf{v}\varepsilon + \frac{\theta^o}{2}\mathbf{p}\mathbf{a}\varepsilon^2\right) = \left(1 + \frac{\theta_0}{2}\mathbf{a}\varepsilon\right)\mathbf{v}\varepsilon = \mathbf{v}\varepsilon = \mathbf{V}$$

As expected, the translation does not affect the free vector.

Screw movement of a poin

It can be shown that

$$\Lambda = RT = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{o}\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^{o}}{2}\varepsilon,$$

and the screw motion of a point can be written as:

$$P' = \Lambda P \Lambda^{\dagger} = RTP(RT)^{\dagger}.$$

It is important that the translation of $\mathbf{T}$ is carried out along the same axis around which the rotation takes place. Then the movement will be screw and the operations R and T commute:

$$RT = TR.$$

The use of separate biquaternions R and T allows for translations and rotations with different axes. Additionally, we will write:

$$\Lambda^{\dagger} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^{o}\varepsilon) + \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\mathbf{a}\right)\frac{\theta^{o}}{2}\varepsilon.$$

Screw motion of line

For a line defined by a screw $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$, the screw motion is set by the same biquaternion Λ, however, the sandwich formula looks somewhat different:

$$\mathbf{L}' = \Lambda\mathbf{L}\Lambda^*,$$

$$\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^o}{2}\varepsilon,$$
$$\Lambda^* = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^o\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} + \sin\frac{\theta}{2}\right)\frac{\theta^o}{2}\varepsilon.$$

This formula can be obtained directly from the transfer principle by replacing quaternions with biquaternions in the sandwich formula for quaternions, since the quaternion conjugation $*$ is used here, and not its combination with the dual $\dagger$.

Pure rotation and pure translation

Consider the pure rotation of a line using a biquaternion R

$$\boxed{\mathbf{L}' = R\mathbf{L}R^*} \quad R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^O\varepsilon), \quad R^* = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^O\varepsilon).$$

It is possible to calculate:

$$\mathbf{L}' = R\mathbf{L}R^* = R\big(\mathbf{v} + \mathbf{m}\varepsilon\big)R^* = \cos\theta\mathbf{v} + \sin\theta\mathbf{a}\times\mathbf{v} + (1-\cos\theta)(\mathbf{a},\mathbf{v})\mathbf{a} + \\ + \Big(\sin\theta\mathbf{a}^O\times\mathbf{v} + (1-\cos\theta)\mathbf{v}\times\mathbf{a}\times\mathbf{a}^O + \cos\theta\mathbf{m} + \sin\theta\mathbf{a}\times\mathbf{m} + (1-\cos\theta)(\mathbf{a},\mathbf{m})\mathbf{a}\Big)\varepsilon$$

Consider the pure translation of a line using a biquaternion T

$$\boxed{\mathbf{L}' = T\mathbf{L}T^*} \quad T = 1 + \frac{\theta^O}{2}\mathbf{a}\varepsilon, \quad T^* = 1 - \frac{\theta^O}{2}\mathbf{a}\varepsilon.$$

It is possible to calculate:

$$\mathbf{L}' = T\mathbf{L}T^* = \mathbf{v} + (\mathbf{m} + \theta^O\mathbf{a}\times\mathbf{v})\varepsilon$$

Screw motion of the plane

For the plane defined by the biquaternion $\Pi = \mathbf{n} + d\varepsilon$, the screw motion is also defined by the biquaternion Λ, and the sandwich formula looks the same as for the point:

$$\Pi' = \Lambda \Pi \Lambda^{\dagger},$$

$$\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{O}\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^{O}}{2}\varepsilon,$$
$$\Lambda^{\dagger} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^{O}\varepsilon) + \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\mathbf{a}\right)\frac{\theta^{O}}{2}\varepsilon.$$

Let's consider separately the rotation of the plane and the translation.

Plane rotation and translation

Consider the pure rotation of the plane $\Pi = \mathbf{n} + d\varepsilon$ using the biquaternion R

$$\Pi^{\dagger} = R\Pi R^{\dagger}, \quad R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{o}\varepsilon), \quad R^{\dagger} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^{o}\varepsilon)$$

It is possible to calculate:

$$\Pi' = R\Pi R^{\dagger} = \cos\theta\mathbf{n} + \sin\theta\mathbf{a}\times\mathbf{n} + (1-\cos\theta)(\mathbf{n},\mathbf{a})\mathbf{a} + \big(d - \sin\theta(\mathbf{a}^{o},\mathbf{n}) - (1-\cos\theta)(\mathbf{a},\mathbf{n},\mathbf{a}^{o})\big)\varepsilon.$$

Consider a pure translation of the plane using a biquaternion$T$

$$\Pi' = T\Pi T^{\dagger}, \quad T = 1 + \frac{\theta^{o}}{2}\mathbf{a}\varepsilon, \quad T^{\dagger} = 1 + \frac{\theta^{o}}{2}\mathbf{a}\varepsilon = T$$

It is possible to calculate:

$$\Pi' = T\Pi T^{\dagger} = \mathbf{n} + (d + \theta^{o}(\mathbf{a},\mathbf{n}))\varepsilon$$

Screw motion using biquaternions

$$\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^{O}\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^{O}}{2}\varepsilon,$$

$$\Lambda^{*} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^{O}\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} + \sin\frac{\theta}{2}\right)\frac{\theta^{O}}{2}\varepsilon,$$

$$\Lambda^{\dagger} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^{O}\varepsilon) + \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\mathbf{a}\right)\frac{\theta^{O}}{2}\varepsilon.$$

- For point $P' = \Lambda P \Lambda^{\dagger}$.
- For line $\mathbf{L}' = \Lambda \mathbf{L} \Lambda^{*}$.
- For plane $\Pi' = \Lambda \Pi \Lambda^{\dagger}$.

Asymptote implementation

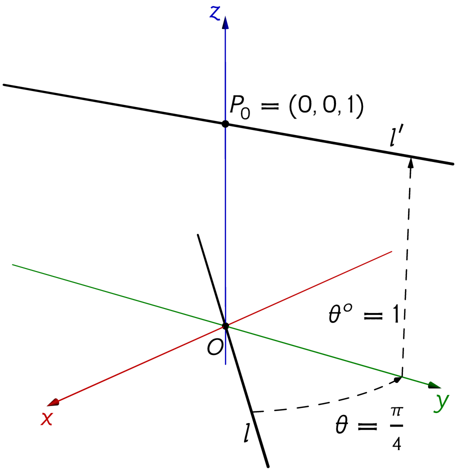
Asymptote language

- Asymptote is a specialized language for vector graphics, both 2D and 3D.
- Has C-like syntax.
- The closest analogue is $PGF/TikZ$, however, Asymptote is imperative, not declarative.
- Was chosen primarily because of the possibility of direct visualization of calculated objects.

Program structure

The main purpose of this work was the implementation of biquaternions in the form of an Asymptote language structure, which, however, is impossible without the implementation of dual numbers and quaternions.
The structure of our small library looks like this:

```
/
├── struct
│   ├── dual.asy
│   ├── quaternion.asy
│   └── dualquaternion.asy
├── test
│   ├── dual.asy
│   ├── quaternion.asy
│   └── dualquaternion.asy
├── img
│   ├── helicoid.asy
│   └── plucker_conoid.asy
└── config.asy
```



Consider the screw motion of a line l with a guide vector $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 1, 0)^T$ and passing through the origin.

$$\mathbf{L} = \mathbf{v} + \mathbf{0}\varepsilon, \text{ т.к. } \mathbf{m} = \mathbf{0} \times \mathbf{v}.$$

The screw axis is Oz and the associated biquaternion is

$$\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon = \mathbf{a}.$$

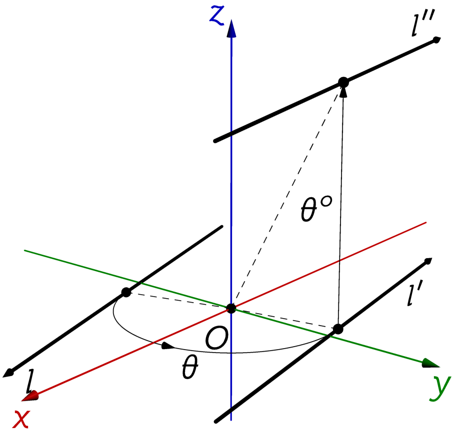
Let's set the dual angle $\Theta = \frac{\pi}{4} + 1$ and the biquaternion of screw motion will be written as:

$$\Lambda = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{a}$$

The screw motion is given by the formula $\mathbf{L}' = \Lambda\mathbf{L}\Lambda^*$, which, after calculations, gives a line

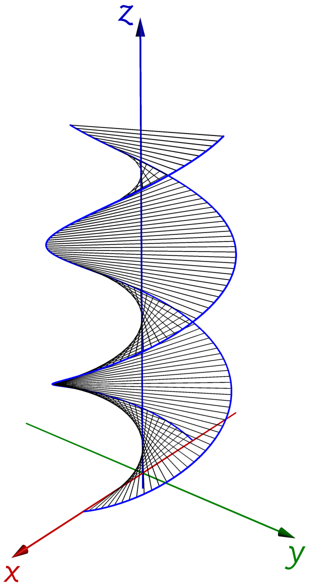
$$\mathbf{L}' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\varepsilon, \quad \mathbf{p}_0 = \mathbf{v}' \times \mathbf{m}'$$

A more complex example of the screw movement of a straight line




```
triple P = (0, -1/2, 0);
triple v = dir(colatitude=90, longitude=15);
triple m = cross(P, v);
DualQuaternion L = screw(v, m);

DualQuaternion A = screw(Z, cross(O, Z));
Dual Theta = Dual(radians(175), 1);

ScrewMotion Rotation =
↪  LineSandwichFormula(Theta=Dual(real(Theta), 0), A=A);
ScrewMotion Translation = LineSandwichFormula(Theta=Dual(0,
↪  dual(Theta)), A=A);
ScrewMotion Motor = LineSandwichFormula(Theta=Theta, A=A);

DualQuaternion L1 = Rotation(L);
DualQuaternion L2 = Translation(L1);
DualQuaternion L2_alt = Motor(L);

triple P0 = cross(vec(L.q), vec(L.qo));
triple P10 = cross(vec(L1.q), vec(L1.qo));
triple P20 = cross(vec(L2.q), vec(L2.qo));
```

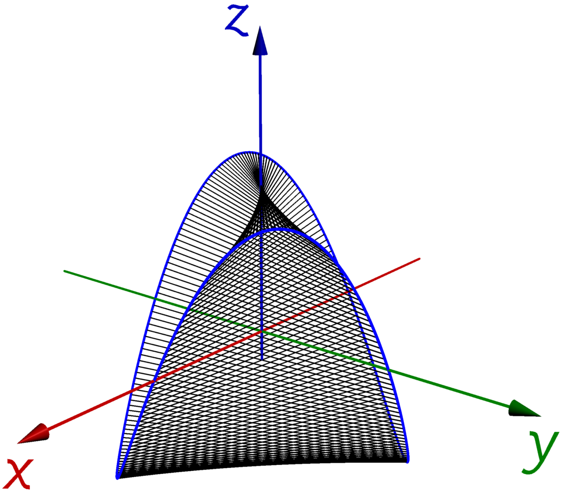
A helicoid produced by screw motion



The helicoid in the figure on the left is obtained by a uniform screw motion of a straight line Ox along the Oz axis. Calculations were performed using biquaternions.

- Screw (pure biquaternion) $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$, where $\mathbf{v} = (1, 0, 0)^T$ и $\mathbf{m} = (0, 0, 0)^T$ represents Ox axis.
- Screw (pure biquaternion) $\mathbf{A} = \mathbf{a} + \mathbf{a}^o$, where $\mathbf{a} = (0, 0, 1)^T$ and $\mathbf{a}^o = (0, 0, 0)^T$ represents the Oz axis, along which the screw movement is carried out.
- Dual angle $\Theta = \dfrac{\pi}{40} + \dfrac{\pi}{80}\varepsilon$.
- The biquaternions $P_1 = 1 + \mathbf{v}\varepsilon$ and $P_2 = 1 - \mathbf{v}\varepsilon$ set the points of the segment, they are also the points of the helix (drawn in blue).
- The unit biquaternion $\Lambda wasconstructed = \cos\dfrac{\Theta}{2} + \sin\dfrac{\Theta}{2}\mathbf{A}$, which was used in sandwich formulas for the screw motion of a straight line $\mathbf{L}' = \Lambda\mathbf{L}\Lambda^*$ and points $P' = \Lambda P \Lambda^\dagger$.
- Repeated application of the sandwich formula made it possible to obtain all the forming surfaces of the helicoid shown in the figure.

Plucker conoid



- The Plucker conoid is obtained by rotating a segment around the Oz axis and simultaneously oscillating along the same axis in the aisles of the segment $[-1, 1]$.
- In this case, it is impossible use a fixed dual angle, so the angle is parametric:

$$\Theta = t + \sin(2t)\varepsilon.$$

- Parameter t takes values from $[0, 2\pi]$ and one screw movement was performed for each of parameter value.
- All the positions of the segment that are obtained visually form a surface.

Conclusion

Results

- Biquaternions lose in computing speed to matrix calculations.
- Just like quaternions, biquaternions are free from the effect of gimbal lock.
  item Biquaternions are easily renormalized, unlike matrices.
- Allows to rotate planes and straight lines as a whole.
- Requires less memory to store the parameters.

The created program makes it easy to manipulate biquaternions and immediately visualize them as points, straight lines and planes.

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