> restart : libname := "C:\\TruncatedSeries2021\\lib", libname :
>

Laurent Case

Example 1 - automatically with ExhaustiveUseConfirmation:

> eq := (-1 + x + x^2 + O(x^3)) * theta(y(x), x, 2) + (-2 + O(x^3)) * theta(y(x), x, 1) + (x + 6 * x^2 + O(x^4)) * y(x);

$$eq := (-1+x+x^2+O(x^3))\,\theta(y(x),x,2) + (-2+O(x^3))\,\theta(y(x),x,1) + (x+6x^2+O(x^4))\,y(x) \quad \textbf{(1.1)}$$

> sol := TruncatedSeries:-LaurentSolution(eq, y(x));

$$sol := \left[\frac{\_c_1}{x^2} - \frac{5\,\_c_1}{x} + \_c_2 + O(x),\ \_c_2 + \frac{x\,\_c_2}{3} + \frac{5x^2\,\_c_2}{6} + \frac{13x^3\,\_c_2}{30} + O(x^4)\right] \quad \textbf{(1.2)}$$

> TruncatedSeries:-ExhaustiveUseConfirmation(sol, eq, y(x), `laurent`);

Equation prolongation #1

$$(-2-x^3+O(x^4))\,\theta(y(x),x,1) + (x^2+x-1-x^3+O(x^4))\,\theta(y(x),x,2) + (6x^2+x-x^4+O(x^5))\,y(x)$$

Additional term(s) in the equation prolongation:

$$y(x)\,(-x^4+O(x^5)) + \theta(y(x),x,1)\,(-x^3+O(x^4)) + \theta(y(x),x,2)\,(-x^3+O(x^4))$$

The equation solution:

$$\left[\frac{\_c_1}{x^2} - \frac{5\,\_c_1}{x} + \_c_2 + x\left(-\frac{37\,\_c_1}{3} + \frac{\_c_2}{3}\right) + O(x^2),\ \_c_2 + \frac{x\,\_c_2}{3} + \frac{5x^2\,\_c_2}{6} + \frac{13x^3\,\_c_2}{30} + \frac{11x^4\,\_c_2}{24} + O(x^5)\right]$$

Additional term(s) in the equation solution:

$$\left[O(x^2) + \frac{(-37\,\_c_1 + \_c_2)\,x}{3},\ \frac{11x^4\,\_c_2}{24} + O(x^5)\right]$$

Equation prolongation #2

$$(-2+x^3+O(x^4))\,\theta(y(x),x,1) + (x^2+x-1+x^3+O(x^4))\,\theta(y(x),x,2) + (6x^2+x+x^4+O(x^5))\,y(x)$$

Additional term(s) in the equation prolongation:

$$y(x)\,(x^4+O(x^5)) + \theta(y(x),x,1)\,(x^3+O(x^4)) + \theta(y(x),x,2)\,(x^3+O(x^4))$$

The equation solution:

$$\left[\frac{\_c_1}{x^2} - \frac{5\,\_c_1}{x} + \_c_2 + x\left(-11\,\_c_1 + \frac{\_c_2}{3}\right) + O(x^2),\ \_c_2 + \frac{x\,\_c_2}{3} + \frac{5x^2\,\_c_2}{6} + \frac{13x^3\,\_c_2}{30} + \frac{43x^4\,\_c_2}{72} + O(x^5)\right]$$

Additional term(s) in the equation solution:

$$\left[O(x^2) + \frac{(-33\,\_c_1 + \_c_2)\,x}{3},\ \frac{43x^4\,\_c_2}{72} + O(x^5)\right] \quad \textbf{(1.3)}$$

>

Example 2 - an arbitrary equation prolomgation may not lead to a solution prolongation:

> eq1 := TruncatedSeries:-ConstructProlongation(theta(y(x), x, 1) * (x^3 + O(x^4)), eq, y(x));

$$eq1 := (-2+x^3+O(x^4))\,\theta(y(x),x,1) + (-1+x+x^2+O(x^3))\,\theta(y(x),x,2) + (x+6x^2 \quad \textbf{(1.4)}$$

$$+ O(x^4) \big) y(x)$$

> $TruncatedSeries\text{:-}LaurentSolution(eq1, y(x));$

$$\left[\frac{\_c_1}{x^2} - \frac{5\,\_c_1}{x} + \_c_2 + O(x), \_c_2 + \frac{x\,\_c_2}{3} + \frac{5\,x^2\,\_c_2}{6} + \frac{13\,x^3\,\_c_2}{30} + O(x^4) \right] \quad (1.5)$$

>

Example 3 - manual step by step with DifferentProlongationExtras:

$$DifferentProlongationExtras \quad (1.6)$$

> $eq := (x + O(x\text{^}2)) * \theta(y(x), x, 1) + O(x\text{^}2) * y(x);$

$$eq := (x + O(x^2))\, \theta(y(x), x, 1) + O(x^2)\, y(x) \quad (1.7)$$

> $sol := TruncatedSeries\text{:-}LaurentSolution(eq, y(x));$

$$sol := [\_c_1 + O(x)] \quad (1.8)$$

> $dp := TruncatedSeries\text{:-}DifferentProlongationExtras(eq, y(x), `laurent`);$

$$dp := [y(x)\,(-x^2 + O(x^3)), y(x)\,(x^2 + O(x^3))] \quad (1.9)$$

> $eq1 := TruncatedSeries\text{:-}ConstructProlongation(dp[1], eq, y(x));$

$$eq1 := (x + O(x^2))\, \theta(y(x), x, 1) + y(x)\,(-x^2 + O(x^3)) \quad (1.10)$$

> $eq2 := TruncatedSeries\text{:-}ConstructProlongation(dp[2], eq, y(x));$

$$eq2 := (x + O(x^2))\, \theta(y(x), x, 1) + y(x)\,(x^2 + O(x^3)) \quad (1.11)$$

> $sol1 := TruncatedSeries\text{:-}LaurentSolution(eq1, y(x));$

$$sol1 := [x\,\_c_1 + \_c_1 + O(x^2)] \quad (1.12)$$

> $sol2 := TruncatedSeries\text{:-}LaurentSolution(eq2, y(x));$

$$sol2 := [-x\,\_c_1 + \_c_1 + O(x^2)] \quad (1.13)$$

>

Regular Case

Example 4 - manual step by step with DifferentProlongationExtras:

> $eq := (-1 + x + x\text{^}2 + O(x\text{^}3)) * \theta(y(x), x, 2) + (-2 + x\text{^}2 + O(x\text{^}3)) * \theta(y(x), x, 1) + O(x\text{^}4) * y(x)$

$$eq := (-1 + x + x^2 + O(x^3))\, \theta(y(x), x, 2) + (-2 + x^2 + O(x^3))\, \theta(y(x), x, 1) + O(x^4)\, y(x) \quad (2.1)$$

> $sol := TruncatedSeries\text{:-}RegularSolution(eq, y(x));$

$$sol := \left[-\frac{\_c_1}{x^2} + \frac{4\,\_c_1}{x} + \_c_2 + O(x) + \ln(x)\,(\_c_1 + O(x^4)), \_c_2 + O(x^4) \right] \quad (2.2)$$

> $dp := TruncatedSeries\text{:-}DifferentProlongationExtras(eq, y(x), `regular`);$

$$dp := [y(x)\,(-x^4 + O(x^5)) + \theta(y(x), x, 1)\,(-x^3 + O(x^4)) + \theta(y(x), x, 2)\,(-x^3 + O(x^4)), y(x)\,(x^4 + O(x^5)) + \theta(y(x), x, 1)\,(x^3 + O(x^4)) + \theta(y(x), x, 2)\,(x^3 + O(x^4))] \quad (2.3)$$

> $eq1 := TruncatedSeries\text{:-}ConstructProlongation(dp[1], eq, y(x));$

$$eq1 := (x^2 - 2 - x^3 + O(x^4))\, \theta(y(x), x, 1) + (x^2 + x - 1 - x^3 + O(x^4))\, \theta(y(x), x, 2) + y(x)\,(-x^4 + O(x^5)) \quad (2.4)$$

> $eq2 := TruncatedSeries\text{:-}ConstructProlongation(dp[2], eq, y(x));$

$$eq2 := (x^2 - 2 + x^3 + O(x^4))\, \theta(y(x), x, 1) + (x^2 + x - 1 + x^3 + O(x^4))\, \theta(y(x), x, 2) + y(x)\,(x^4 + O(x^5)) \quad (2.5)$$

> $sol1 := TruncatedSeries\text{:-}RegularSolution(eq1, y(x));$

$$sol1 := \left[-\frac{\_c_1}{x^2} + \frac{4\,\_c_1}{x} + \_c_2 + \frac{2\,x\,\_c_1}{3} + O(x^2) + \ln(x)\left(\_c_1 - \frac{x^4\,\_c_1}{24} + O(x^5)\right), \_c_2 - \frac{x^4\,\_c_2}{24} \quad (2.6)$$

$$+ O(x^5)\Big]$$

> sol2 := TruncatedSeries:-RegularSolution(eq2, y(x));

$$sol2 := \left[-\frac{\_c_1}{x^2} + \frac{4\,\_c_1}{x} + \_c_2 - \frac{2\,x\,\_c_1}{3} + O(x^2) + \ln(x)\left(\_c_1 + \frac{x^4\,\_c_1}{24} + O(x^5)\right), \_c_2 + \frac{x^4\,\_c_2}{24} + O(x^5) \right] \quad (2.7)$$

>

Example 5 - automatically with ExhaustiveUseConfirmation:

> eq := (1 + x^2 + O(x^3)) * theta(y(x), x, 3) + (4-x + 1/2 * x^2 + O(x^3)) * theta(y(x), x, 2) + (4-2 * x + x^2 + O(x^3)) * theta(y(x), x, 1) + O(x^3) * y(x);

$$eq := (1 + x^2 + O(x^3))\,\theta(y(x), x, 3) + \left(4 - x + \frac{x^2}{2} + O(x^3)\right)\theta(y(x), x, 2) + (4 - 2\,x + x^2 + O(x^3))\,\theta(y(x), x, 1) + O(x^3)\,y(x) \quad (2.8)$$

> sol := TruncatedSeries:-RegularSolution(eq, y(x));

$$sol := \left[\frac{\frac{21\,\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + O(x) + \ln(x)\left(\frac{\_c_1}{2\,x^2} + \_c_2 + O(x)\right) + \ln(x)^2\left(\frac{\_c_1}{2} + O(x^3)\right), \frac{\_c_2}{2\,x^2} + \_c_3 + O(x) + \ln(x)\left(\_c_2 + O(x^3)\right), \_c_3 + O(x^3) \right] \quad (2.9)$$

> ExhaustiveUseConfirmation(sol, eq, y(x), `regular`);

Equation prolongation #1

$$(x^2 - 2\,x + 4 - x^3 + O(x^4))\,\theta(y(x), x, 1) + \left(4 - x + \frac{x^2}{2} - x^3 + O(x^4)\right)\theta(y(x), x, 2) + (x^2 + 1 - x^3 + O(x^4))\,\theta(y(x), x, 3) + y(x)\,(-x^3 + O(x^4))$$

Additional term(s) in the equation prolongation:

$$y(x)\,(-x^3 + O(x^4)) + \theta(y(x), x, 1)\,(-x^3 + O(x^4)) + \theta(y(x), x, 2)\,(-x^3 + O(x^4)) + \theta(y(x), x, 3)\,(-x^3 + O(x^4))$$

The equation solution:

$$\left[\frac{\frac{21\,\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + x\left(\frac{61\,\_c_1}{432} - \frac{\_c_2}{18}\right) + O(x^2) + \ln(x)\left(\frac{\_c_1}{2\,x^2} + \_c_2 - \frac{x\,\_c_1}{18} + O(x^2)\right) + \ln(x)^2\left(\frac{\_c_1}{2} + \frac{x^3\,\_c_1}{150} + O(x^4)\right), \frac{\_c_2}{2\,x^2} + \_c_3 - \frac{x\,\_c_2}{18} + O(x^2) + \ln(x)\left(\_c_2 + \frac{x^3\,\_c_2}{75} + O(x^4)\right), \_c_3 + \frac{x^3\,\_c_3}{75} + O(x^4) \right]$$

Additional term(s) in the equation solution:

$$\left[\left(\frac{x^3\,\_c_1}{150} + O(x^4)\right)\ln(x)^2 + \left(-\frac{x\,\_c_1}{18} + O(x^2)\right)\ln(x) + \frac{61\,x\,\_c_1}{432} - \frac{x\,\_c_2}{18} + O(x^2), \left(\frac{x^3\,\_c_2}{75}\right.\right.$$

$$+ O(x^4) \Big) \ln(x) - \frac{x\_c_2}{18} + O(x^2), \frac{x^3\_c_3}{75} + O(x^4) \Big]$$

Equation prolongation #2

$$(x^2 - 2x + 4 + x^3 + O(x^4))\, \theta(y(x), x, 1) + \left(4 - x + \frac{x^2}{2} + x^3 + O(x^4)\right) \theta(y(x), x, 2) + (x^2 + 1 + x^3 + O(x^4))\, \theta(y(x), x, 3) + y(x)\,(x^3 + O(x^4))$$

Additional term(s) in the equation prolongation:

$$y(x)\,(x^3 + O(x^4)) + \theta(y(x), x, 1)\,(x^3 + O(x^4)) + \theta(y(x), x, 2)\,(x^3 + O(x^4)) + \theta(y(x), x, 3)\,(x^3 + O(x^4))$$

The equation solution:

$$\left[\frac{\frac{21\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + x\left(-\frac{47\_c_1}{144} + \frac{\_c_2}{2}\right) + O(x^2) + \ln(x)\left(\frac{\_c_1}{2x^2} + \_c_2 + \frac{x\_c_1}{2} + O(x^2)\right) + \ln(x)^2\left(\frac{\_c_1}{2} - \frac{x^3\_c_1}{150} + O(x^4)\right), \frac{\_c_2}{2x^2} + \_c_3 + \frac{x\_c_2}{2} + O(x^2) + \ln(x)\left(\_c_2 - \frac{x^3\_c_2}{75} + O(x^4)\right), \_c_3 - \frac{x^3\_c_3}{75} + O(x^4)\right]$$

Additional term(s) in the equation solution:

$$\left[\left(-\frac{x^3\_c_1}{150} + O(x^4)\right)\ln(x)^2 + \left(O(x^2) + \frac{x\_c_1}{2}\right)\ln(x) + O(x^2) - \frac{x\left(\frac{1175\_c_1}{24} - 75\_c_2\right)}{150}, \left(-\frac{x^3\_c_2}{75} + O(x^4)\right)\ln(x) + \frac{x\_c_2}{2} + O(x^2), -\frac{x^3\_c_3}{75} + O(x^4)\right] \quad \textbf{(2.10)}$$

\>

Different Logarithm Degree Solutions Prolongations

Example 6 - different prolongations with different logarithm degrees in solutions:

```
> eq := (-1 + x + O(x^2))*theta(y(x), x, 2) + (-2 + x^2 + O(x^3))*theta(y(x), x, 1) + O(x^4)*y(x)
```

$$eq := (-1 + x + O(x^2))\, \theta(y(x), x, 2) + (-2 + x^2 + O(x^3))\, \theta(y(x), x, 1) + O(x^4)\, y(x) \quad \textbf{(3.1)}$$

```
> sol := TruncatedSeries:-RegularSolution(eq, y(x));
```

$$sol := [\_c_1 + O(x^4)] \quad \textbf{(3.2)}$$

```
> dp := TruncatedSeries:-DifferentLnDegreeExtras(eq, y(x));
```

$$dp := \left[\theta(y(x), x, 2)\left(\frac{3x^2}{2} + O(x^3)\right), \theta(y(x), x, 2)\,(2x^2 + O(x^3))\right] \quad \textbf{(3.3)}$$

```
> eq1 := TruncatedSeries:-ConstructProlongation(dp[1], eq, y(x));
```

$$eq1 := (-2 + x^2 + O(x^3))\, \theta(y(x), x, 1) + \left(-1 + x + \frac{3x^2}{2} + O(x^3)\right) \theta(y(x), x, 2) + O(x^4)\, y(x) \quad \textbf{(3.4)}$$

```
> eq2 := TruncatedSeries:-ConstructProlongation(dp[2], eq, y(x));
```

$$eq2 := (-2 + x^2 + O(x^3))\, \theta(y(x), x, 1) + (-1 + x + 2x^2 + O(x^3))\, \theta(y(x), x, 2) + O(x^4)\, y(x) \quad \textbf{(3.5)}$$

```
> sol1 := TruncatedSeries:-RegularSolution(eq1, y(x));
```

(3.6)

$$soll := \left[\frac{\_c_1}{x^2} - \frac{4\,\_c_1}{x} + \_c_2 + O(x),\ \_c_2 + O(x^4) \right] \quad (3.6)$$

> *sol2* := *TruncatedSeries:-RegularSolution*(*eq2*, *y*(*x*));

$$sol2 := \left[\frac{\_c_1}{x^2} - \frac{4\,\_c_1}{x} + \_c_2 + O(x) + \ln(x)\left(\_c_1 + O(x^4)\right),\ \_c_2 + O(x^4) \right] \quad (3.7)$$

>

Example 7 - no different prolongations with different logarithm degrees in solutions:

> *TruncatedSeries:-DifferentLnDegreeExtras*(*eq1*, *y*(*x*));

$$[\ ] \quad (3.8)$$

> *TruncatedSeries:-DifferentLnDegreeExtras*(*eq2*, *y*(*x*));

$$[\ ] \quad (3.9)$$

Example 8 - different prolongations with different logarithm degrees in solutions with higher degrees of logarithms:

> *eq* := (1 + O(x^2)) * theta(*y*(*x*), *x*, 3) + (4 + O(*x*)) * theta(*y*(*x*), *x*, 2) + (4 − 2 * *x* + O(x^2)) * theta(*y*(*x*), *x*, 1) + O(x^2) * *y*(*x*);

$$eq := \left(1 + O(x^2)\right)\theta(y(x), x, 3) + (4 + O(x))\,\theta(y(x), x, 2) + \left(4 - 2x + O(x^2)\right)\theta(y(x), x, 1) + O(x^2)\,y(x) \quad (3.10)$$

> *sol* := *TruncatedSeries:-RegularSolution*(*eq*, *y*(*x*));

$$sol := \left[\_c_1 + O(x^2)\right] \quad (3.11)$$

> *dp* := *TruncatedSeries:-DifferentLnDegreeExtras*(*eq*, *y*(*x*));

$$dp := \left[y(x)\left(-8x^2 + O(x^3)\right) + O(x^3)\,\theta(y(x), x, 1) + \theta(y(x), x, 2)\,O(x^3) + \theta(y(x), x, 3)\,O(x^3),\ y(x)\left(-7x^2 + O(x^3)\right) + \theta(y(x), x, 1)\left(x^2 + O(x^3)\right) + \theta(y(x), x, 2)\left(x^2 + x + O(x^3)\right) + \theta(y(x), x, 3)\left(x^2 + O(x^3)\right)\right] \quad (3.12)$$

> *eq1* := *TruncatedSeries:-ConstructProlongation*(*dp*[1], *eq*, *y*(*x*));

$$eq1 := \left(4 - 2x + O(x^3)\right)\theta(y(x), x, 1) + \left(4 + O(x^3)\right)\theta(y(x), x, 2) + \left(1 + O(x^3)\right)\theta(y(x), x, 3) + y(x)\left(-8x^2 + O(x^3)\right) \quad (3.13)$$

> *eq2* := *TruncatedSeries:-ConstructProlongation*(*dp*[2], *eq*, *y*(*x*));

$$eq2 := \left(4 - 2x + x^2 + O(x^3)\right)\theta(y(x), x, 1) + \left(4 + x^2 + x + O(x^3)\right)\theta(y(x), x, 2) + \left(1 + x^2 + O(x^3)\right)\theta(y(x), x, 3) + y(x)\left(-7x^2 + O(x^3)\right) \quad (3.14)$$

> *sol1* := *TruncatedSeries:-RegularSolution*(*eq1*, *y*(*x*));

$$sol1 := \left[\frac{\_c_2}{x^2} + \frac{4\,\_c_2 - \frac{6\,\_c_1}{5}}{x} + \_c_3 + O(x) + \ln(x)\left(\frac{\_c_1}{5x^2} + \frac{4\,\_c_1}{5x} + \_c_1 + O(x)\right),\ \frac{\_c_2}{x^2} + \frac{4\,\_c_2}{x} + \_c_3 + O(x),\ -\frac{6\,\_c_1}{5x} + \_c_3 + O(x) + \ln(x)\left(\frac{\_c_1}{5x^2} + \frac{4\,\_c_1}{5x} + \_c_1 + O(x)\right),\ \_c_3 + \frac{x^2\,\_c_3}{4} + O(x^3) \right] \quad (3.15)$$

> *sol2* := *TruncatedSeries:-RegularSolution*(*eq2*, *y*(*x*));

$$sol2 := \left[\frac{-\frac{304\,\_c_1}{121} - \frac{4\,\_c_2}{11}}{x^2} + \frac{-\frac{1816\,\_c_1}{121} - \frac{32\,\_c_2}{11}}{x} + \_c_3 + O(x) + \ln(x)\left(-\frac{4\,\_c_1}{11x^2} - \frac{32\,\_c_1}{11x} + \_c_2 + O(x)\right) + \ln(x)^2\left(\frac{\_c_1}{2} + \frac{7x^2\,\_c_1}{64} + O(x^3)\right),\ -\frac{4\,\_c_2}{11x^2} - \frac{32\,\_c_2}{11x} + \_c_3 + O(x) \right. \quad (3.16)$$

$$+ \ln(x)\left(\_c_2 + \frac{7x^2\,\_c_2}{32} + O(x^3)\right), \_c_3 + \frac{7x^2\,\_c_3}{32} + O(x^3)\Bigg]$$

>