> restart;
>

Load TruncatedSeries2021.zip from http://www.ccas.ru/ca/_media/truncatedseries2021.zip
This archive includes two files: maple.ind and maple.lib.
Put these files to some directory, for example to "/usr/userlib"

> libname := "/usr/userlib", libname :
>
> with(TruncatedSeries) :
>
>

A single series in a solution

Consider the following simple equation:

> $eq := \left(1 + 3\,x + \mathrm{O}\left(x^3\right)\right) y(x) + \left(x^3 + \frac{x^5}{3} + \mathrm{O}\left(x^6\right)\right) \left(\frac{\mathrm{d}}{\mathrm{d}x}\, y(x)\right) :$

Using the FormalSolution command we obtain exponential-logarithmic solutions whose regular part is calculated to the maximum possible degree:

> $FormalSolution(eq, y(x));$

$$\left[\mathrm{e}^{\frac{1}{2x^2} + \frac{3}{x}}\, x^{1/3} \left(\_c_1 + \mathrm{O}(x)\right)\right] \tag{1.1}$$

If, when calling the FormalSolution command, the optional argument 'output' = 'literal' is used, then the regular part of the solution is calculated to the maximum degree and, in addition, a term is added with coefficients depending on some of literals.

> $FormalSolution(eq, y(x), 'output'='literal')$

$$\mathrm{e}^{\frac{1}{2x^2} + \frac{3}{x}}\, x^{1/3} \left(\_c_1 + \left(-U_{[0,\,3]}\,\_c_1 + U_{[1,\,6]}\,\_c_1 + \_c_1\right) x + \mathrm{O}\left(x^2\right)\right) \tag{1.2}$$

As a result of running the FormalSolution command with the optional argument 'counterexample' = 'Eqs', the variable Eqs will be assigned a pair of the equations which forms one of the possible counterexamples:

> $FormalSolution(eq, y(x), 'counterexample'='Eqs') :$
>

For the first equation of the counterexample using FormalSolution we obtain a truncated solution

> $Eqs[1]$

$$\left(1 + 3\,x + x^3 + \mathrm{O}\left(x^4\right)\right) y(x) + \left(x^3 + \frac{x^5}{3} + 4\,x^6 + \mathrm{O}\left(x^7\right)\right) \left(\frac{\mathrm{d}}{\mathrm{d}x}\, y(x)\right) = 0 \tag{1.3}$$

> $FormalSolution(Eqs[1], y(x))$

$$\left[\mathrm{e}^{\frac{1}{2x^2} + \frac{3}{x}}\, x^{1/3} \left(\_c_1 + 4\,\_c_1\, x + \mathrm{O}\left(x^2\right)\right)\right] \tag{1.4}$$

And for the second equation of the counterexample we obtain

> $Eqs[2]$

$$\left(1+3x+O\left(x^4\right)\right)y(x)+\left(x^3+\frac{x^5}{3}-4x^6+O\left(x^7\right)\right)\left(\frac{d}{dx}y(x)\right)=0 \quad (1.5)$$

> $FormalSolution(Eqs[2], y(x))$

$$\left[e^{\frac{1}{2x^2}+\frac{3}{x}}x^{1/3}\left(\_c_1-3\_c_1 x+O\left(x^2\right)\right)\right] \quad (1.6)$$

One can see that **(1.4)** and **(1.6)** are prolongations of **(1.1)**, they differ in all regular parts.

>

>

A solution containing two series: a power series and a series in the regular part

Consider the 2-order equation:

> $eq := O\left(x^{10}\right)y(x)+\left(1+3x+O\left(x^3\right)\right)\left(\frac{d}{dx}y(x)\right)+\left(x^3+\frac{x^5}{3}+O\left(x^6\right)\right)\left(\frac{d^2}{dx^2}y(x)\right):$

Using the FormalSolution command we obtain exponential-logarithmic solutions whose regular parts are calculated to the maximum possible degrees:

> $FormalSolution(eq, y(x));$

$$\left[\_c_1+O\left(x^{11}\right)+e^{\frac{1}{2x^2}+\frac{3}{x}}x^{10/3}\left(\_c_2+O(x)\right)\right] \quad (2.1)$$

If, when calling the FormalSolution command, the optional argument 'output' = 'literal' is used, then the regular parts of the solution are calculated to the maximum degree and, in addition, terms are added with coefficients depending on some of literals.

> $FormalSolution(eq, y(x), 'output'='literal')$

$$\_c_1-\frac{U_{[0,10]}\_c_1 x^{11}}{11}+O\left(x^{12}\right)+e^{\frac{1}{2x^2}+\frac{3}{x}}x^{10/3}\left(\_c_2+\left(-U_{[1,3]}\_c_2+U_{[2,6]}\_c_2-2\_c_2\right)x+O\left(x^2\right)\right) \quad (2.2)$$

As a result of running the FormalSolution command with the optional argument 'counterexample' = 'Eqs', the variable Eqs will be assigned a pair of the equations which forms one of the possible counterexamples:

> $FormalSolution(eq, y(x), 'counterexample'='Eqs'):$

>

For the first equation of the counterexample using FormalSolution we obtain a truncated solution

> $Eqs[1]$

(2.3)

$$\left(5x^{10}+O(x^{11})\right)y(x)+\left(1+3x-2x^3+O(x^4)\right)\left(\frac{d}{dx}y(x)\right)+\left(x^3+\frac{x^5}{3}+O(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)=0 \quad \textbf{(2.3)}$$

> *FormalSolution*(*Eqs*[1], $y(x)$)

$$\left[\_c_1-\frac{5\_c_1x^{11}}{11}+O(x^{12})+e^{\frac{1}{2x^2}+\frac{3}{x}}x^{10/3}\left(\_c_2+O(x^2)\right)\right] \quad \textbf{(2.4)}$$

And for the second equation of the counterexample we obtain

> *Eqs*[2]

$$\left(-x^{10}+O(x^{11})\right)y(x)+\left(1+3x+5x^3+O(x^4)\right)\left(\frac{d}{dx}y(x)\right)+\left(x^3+\frac{x^5}{3}-5x^6+O(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)=0 \quad \textbf{(2.5)}$$

> *FormalSolution*(*Eqs*[2], $y(x)$)

$$\left[\_c_1+\frac{\_c_1x^{11}}{11}+O(x^{12})+e^{\frac{1}{2x^2}+\frac{3}{x}}x^{10/3}\left(\_c_2-12\_c_2x+O(x^2)\right)\right] \quad \textbf{(2.6)}$$

One can see that **(2.4)** and **(2.6)** are prolongations of **(2.1)**, they differ in all regular parts.

>

>

>

Logarithm in a solution

Consider the equation with full defined coefficients:

> $eq\_full := (1+3x)\,y(x)+\left(2x^3+6x^4+\frac{11x^5}{3}+9x^6\right)\left(\frac{d}{dx}y(x)\right)+\left(x^6+3x^7+\frac{2x^8}{3}-\frac{23x^{10}}{9}+\frac{61x^{12}}{9}\right)\left(\frac{d^2}{dx^2}y(x)\right):$

All its formal solutions can be constructed with any given truncation degree. The truncation degree *k* is set by the optional argument 'top'=*k*. All solutions can be presented by one expression with arbitrary constants $\_c_1$, $\_c_2$, etc. For the truncation degree 1

> *FormalSolution*(*eq_full*, $y(x)$,'*output*'='*compact*', '*top*'= 1);

$$e^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2+\frac{\_c_1}{x^2}-\frac{5\_c_1}{3x}+\left(-\frac{7\_c_2}{9}-\frac{292\_c_1}{81}\right)x+O(x^2)+\ln(x)\left(\_c_1-\frac{7\_c_1x}{9}+O(x^2)\right)\right) \quad \textbf{(3.1)}$$

for the truncation degree 3

> *FormalSolution*(*eq_full*, $y(x)$,'*output*'='*compact*', '*top*'= 3);

$$e^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2+\frac{\_c_1}{x^2}-\frac{5\_c_1}{3x}+\left(-\frac{7\_c_2}{9}-\frac{292\_c_1}{81}\right)x+\left(\frac{131\_c_2}{108}+\frac{7397\_c_1}{3888}\right)x^2+\left(-\frac{3451\_c_2}{1620}+\frac{42517\_c_1}{97200}\right)x^3+O(x^4)+\ln(x)\left(\_c_1-\frac{7\_c_1x}{9}+\frac{131\_c_1x^2}{108}-\frac{3451\_c_1x^3}{1620}\right.\right. \quad \textbf{(3.2)}$$

$$+\mathrm{O}(x^4)\Big)\Big)$$

Consider the truncated equation:

> $eq := (1+3x+\mathrm{O}(x^7))\,y(x) + \left(2x^3+6x^4+\frac{11x^5}{3}+9x^6+\mathrm{O}(x^{10})\right)\left(\frac{d}{dx}y(x)\right) + \left(x^6+3x^7 + \frac{2x^8}{3} - \frac{23x^{10}}{9} + \frac{61x^{12}}{9} + \mathrm{O}(x^{13})\right)\left(\frac{d^2}{dx^2}y(x)\right):$

The general solution for equation with all truncated coefficient is presented by the list of several expressions with arbitrary constants. The general solution generates a particular solution with maximum truncation degrees, if the substitution of the corresponding values of arbitrary constants does not change the structure of the solution and does not change the valuations of the series included in the general solution; otherwise, the substitution may give a particular solution with a degree of truncation that is not maximum:

> $FormalSolution(eq, y(x));$

$$\left[\mathrm{e}^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2+\frac{\_c_1}{x^2}-\frac{5\,\_c_1}{3x}+\mathrm{O}(x)+\ln(x)\left(\_c_1-\frac{7\,\_c_1 x}{9}+\frac{131\,\_c_1 x^2}{108}+\mathrm{O}(x^3)\right)\right), \mathrm{e}^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2-\frac{7\,\_c_2 x}{9}+\frac{131\,\_c_2 x^2}{108}+\mathrm{O}(x^3)\right)\right] \quad \textbf{(3.3)}$$

If, when calling the FormalSolution command, the optional argument 'output' = 'literal' is used, then the regular parts of the solution are calculated to the maximum degree and, in addition, terms are added with coefficients depending on some of literals.

> $FormalSolution(eq, y(x), 'output'='literal')$

$$\mathrm{e}^{\frac{1}{2x^2}}x^{1/3}\Bigg(\_c_2+\frac{\_c_1}{x^2}-\frac{5\,\_c_1}{3x}+\left(-\frac{7}{9}\_c_2-\frac{292}{81}\_c_1-\frac{1}{3}\_c_1U_{[0,7]}+\frac{1}{3}\_c_1U_{[1,10]}-\frac{1}{3}\_c_1U_{[2,13]}\right)x+\left(\frac{131}{108}\_c_2+\frac{7397}{3888}\_c_1+\frac{55}{72}\_c_1U_{[0,7]}-\frac{55}{72}\_c_1U_{[1,10]}+\frac{55}{72}\_c_1U_{[2,13]}-\frac{1}{8}\_c_1U_{[0,8]}+\frac{1}{8}\_c_1U_{[1,11]}-\frac{1}{8}\_c_1U_{[2,14]}\right)x^2+\left(-\frac{3451}{1620}\_c_2+\frac{42517}{97200}\_c_1-\frac{7403}{5400}\_c_1U_{[0,7]}+\frac{8003}{5400}\_c_1U_{[1,10]}-\frac{9683}{5400}\_c_1U_{[2,13]}+\frac{131}{360}\_c_1U_{[0,8]}-\frac{131}{360}\_c_1U_{[1,11]}+\frac{131}{360}\_c_1U_{[2,14]}-\frac{1}{15}\_c_2U_{[0,7]}+\frac{1}{15}\_c_2U_{[1,10]}-\frac{1}{15}\_c_2U_{[2,13]}-\frac{1}{15}\_c_1U_{[0,9]}+\frac{1}{15}\_c_1U_{[1,12]}-\frac{1}{15}\_c_1U_{[2,15]}\right)x^3+\mathrm{O}(x^4)+\ln(x)\left(\_c_1-\frac{7\,\_c_1x}{9}+\frac{131\,\_c_1x^2}{108}+\left(-\frac{3451}{1620}\_c_1-\frac{1}{15}\_c_1U_{[0,7]}+\frac{1}{15}\_c_1U_{[1,10]}-\frac{1}{15}\_c_1U_{[2,13]}\right)x^3+\mathrm{O}(x^4)\right)\Bigg) \quad \textbf{(3.4)}$$

As a result of running the FormalSolution command with the optional argument 'counterexample' = 'Eqs', the variable Eqs will be assigned a pair of the equations which forms one of the possible counterexamples:

> $FormalSolution(eq, y(x), 'counterexample'='Eqs')$:

>

For the first equation of the counterexample using FormalSolution we obtain a truncated solution

> $Eqs[1]$

$$\left(1+3x+3x^7+O(x^8)\right)y(x)+\left(6x^4+2x^3+\frac{11x^5}{3}+9x^6+4x^{10}+O(x^{11})\right)\left(\frac{d}{dx}y(x)\right)+\left(3x^7+x^6+\frac{2x^8}{3}-\frac{23x^{10}}{9}+\frac{61x^{12}}{9}-4x^{13}+O(x^{14})\right)\left(\frac{d^2}{dx^2}y(x)\right)=0 \quad (3.5)$$

> $FormalSolution(Eqs[1], y(x))$

$$\left[e^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2+\frac{\_c_1}{x^2}-\frac{5\_c_1}{3x}+\left(-\frac{7\_c_2}{9}-\frac{157\_c_1}{81}\right)x+O(x^2)+\ln(x)\left(\_c_1-\frac{7\_c_1x}{9}+\frac{131\_c_1x^2}{108}-\frac{2911\_c_1x^3}{1620}+O(x^4)\right)\right), e^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2-\frac{7\_c_2x}{9}+\frac{131\_c_2x^2}{108}-\frac{2911\_c_2x^3}{1620}+O(x^4)\right)\right] \quad (3.6)$$

And for the second equation of the counterexample we obtain

> $Eqs[2]$

$$\left(1+3x+5x^7+O(x^8)\right)y(x)+\left(6x^4+2x^3+\frac{11x^5}{3}+9x^6+2x^{10}+O(x^{11})\right)\left(\frac{d}{dx}y(x)\right)+\left(3x^7+x^6+\frac{2x^8}{3}-\frac{23x^{10}}{9}+\frac{61x^{12}}{9}+4x^{13}+O(x^{14})\right)\left(\frac{d^2}{dx^2}y(x)\right)=0 \quad (3.7)$$

> $FormalSolution(Eqs[2], y(x))$

$$\left[e^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2+\frac{\_c_1}{x^2}-\frac{5\_c_1}{3x}+\left(-\frac{7\_c_2}{9}-\frac{481\_c_1}{81}\right)x+O(x^2)+\ln(x)\left(\_c_1-\frac{7\_c_1x}{9}+\frac{131\_c_1x^2}{108}-\frac{4207\_c_1x^3}{1620}+O(x^4)\right)\right), e^{\frac{1}{2x^2}}x^{1/3}\left(\_c_2-\frac{7\_c_2x}{9}+\frac{131\_c_2x^2}{108}-\frac{4207\_c_2x^3}{1620}+O(x^4)\right)\right] \quad (3.8)$$

One can see that **(3.6)** and **(3.8)** are prolongations of **(3.3)**, they differ in all series.

>

>

>

Power series solution, irregular solution with truncated regular part and

irregular solution with unknown exponent λ

Consider the third-order equation with coefficients truncated to different degrees:

> $eq := O(x^5)\,y(x) + (3x^4 + 2x^3 + 4x^2 + x + O(x^5))\left(\frac{d}{dx}\,y(x)\right) + (3x^6 + 3x^3 + O(x^7))\left(\frac{d^2}{dx^2}\,y(x)\right) + (x^7 + O(x^{10}))\left(\frac{d^3}{dx^3}\,y(x)\right):$

>

Using the FormalSolution command we obtain exponential-logarithmic solutions whose regular parts are calculated to the maximum possible degrees:

> $FormalSolution(eq, y(x));$

$$\left[\_c_1 + O(x^5) + e^{\frac{1}{3x}}\,x^{2/3}\left(\_c_2 + \frac{35\,\_c_2\,x}{27} + \frac{8947\,\_c_2\,x^2}{1458} + O(x^3)\right) + e^{\frac{1}{x^3} - \frac{1}{3x}}\,y_{reg}(x)\right] \quad \textbf{(4.1)}$$

If, when calling the FormalSolution command, the optional argument 'output' = 'literal' is used, then the regular parts of the solution are calculated to the maximum degree and, in addition, terms are added with coefficients depending on some of literals. In some cases it is possible to obtain the expression for λ which also depends on literals.

> $FormalSolution(eq, y(x), 'output'='literal')$

$$\_c_1 - \frac{\_c_1\,U_{[0,5]}\,x^5}{5} + O(x^6) + e^{\frac{1}{3x}}\,x^{2/3}\left(\_c_2 + \frac{35\,\_c_2\,x}{27} + \frac{8947\,\_c_2\,x^2}{1458} + \left(\frac{5845553}{118098}\,\_c_2 - \frac{1}{9}\,\_c_2\,U_{[1,5]} + \frac{1}{27}\,\_c_2\,U_{[2,7]}\right)x^3 + O(x^4)\right) + e^{\frac{1}{x^3} - \frac{1}{3x}}\,x^{\frac{19}{3} + 3\,U_{[3,10]}}\left(\_c_3 + O(x)\right) \quad \textbf{(4.2)}$$

As a result of running the FormalSolution command with the optional argument 'counterexample' = 'Eqs', the variable Eqs will be assigned a pair of the equations which forms one of the possible counterexamples:

> $FormalSolution(eq, y(x), 'counterexample'='Eqs'):$

>

For the first equation of the counterexample using FormalSolution we obtain a truncated solution

> $Eqs[1]$

$$(5x^5 + O(x^6))\,y(x) + (3x^4 + 2x^3 + 4x^2 + x - 5x^5 + O(x^6))\left(\frac{d}{dx}\,y(x)\right) + (3x^6 + 3x^3 + O(x^8))\left(\frac{d^2}{dx^2}\,y(x)\right) + (x^7 + x^{10} + O(x^{11}))\left(\frac{d^3}{dx^3}\,y(x)\right) = 0 \quad \textbf{(4.3)}$$

> $FormalSolution(Eqs[1], y(x))$

$$\left[-\_c_1\,x^5 + \_c_1 + O(x^6) + e^{\frac{1}{3x}}\,x^{2/3}\left(\_c_2 + \frac{35\,\_c_2\,x}{27} + \frac{8947\,\_c_2\,x^2}{1458} + \frac{5911163\,\_c_2\,x^3}{118098} + O(x^4)\right) + e^{\frac{1}{x^3} - \frac{1}{3x}}\,x^{28/3}\left(\_c_3 + O(x)\right)\right] \quad \textbf{(4.4)}$$

And for the second equation of the counterexample we obtain

> $Eqs[2]$

$$\left(x^5+O(x^6)\right)y(x)+\left(3x^4+2x^3+4x^2+x-3x^5+O(x^6)\right)\left(\frac{d}{dx}y(x)\right)+\left(3x^6+3x^3-3x^7+O(x^8)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(x^7-x^{10}+O(x^{11})\right)\left(\frac{d^3}{dx^3}y(x)\right)=0 \quad \textbf{(4.5)}$$

> $FormalSolution(Eqs[2], y(x))$

$$\left[\_c_1-\frac{\_c_1 x^5}{5}+O(x^6)+e^{\frac{1}{3x}}x^{2/3}\left(\_c_2+\frac{35\,\_c_2 x}{27}+\frac{8947\,\_c_2 x^2}{1458}+\frac{5871797\,\_c_2 x^3}{118098}+O(x^4)\right)+e^{\frac{1}{x^3}-\frac{1}{3x}}x^{10/3}\left(\_c_3+O(x)\right)\right] \quad \textbf{(4.6)}$$

One can see that **(4.4)** and **(4.6)** are prolongations of **(4.1)**, they differ in all regular parts. The exponents λ of the third regular part are also different.

>

>

θ-form of the previous equation

By definition:

> $\theta(y(x), x, 1) = x \cdot diff(y(x), x)$

$$\theta(y(x), x, 1)=x\left(\frac{d}{dx}y(x)\right) \quad \textbf{(5.1)}$$

> $eq := \left(x^4+O(x^7)\right)\theta(y(x),x,3)+\left(3x+O(x^5)\right)\theta(y(x),x,2)+\left(1+3x^3+2x^2+x+O(x^4)\right)\theta(y(x),x,1)+O(x^5)\,y(x)=0;$

$$eq:=\left(x^4+O(x^7)\right)\theta(y(x),x,3)+\left(3x+O(x^5)\right)\theta(y(x),x,2)+\left(3x^3+2x^2+x+1+O(x^4)\right)\theta(y(x),x,1)+O(x^5)\,y(x)=0 \quad \textbf{(5.2)}$$

> $FormalSolution(eq, y(x))$

$$\left[\_c_1+O(x^5)+e^{\frac{1}{3x}}x^{2/3}\left(\_c_2+\frac{35\,\_c_2 x}{27}+\frac{8947\,\_c_2 x^2}{1458}+O(x^3)\right)+e^{\frac{1}{x^3}-\frac{1}{3x}}y_{reg}(x)\right] \quad \textbf{(5.3)}$$

> $FormalSolution(eq, y(x), 'output'='literal')$

$$\_c_1-\frac{\_c_1 U_{[0,5]}x^5}{5}+O(x^6)+e^{\frac{1}{3x}}x^{2/3}\left(\_c_2+\frac{35\,\_c_2 x}{27}+\frac{8947\,\_c_2 x^2}{1458}+\left(\frac{5832431}{118098}\_c_2-\frac{1}{9}\_c_2 U_{[1,4]}+\frac{1}{27}\_c_2 U_{[2,5]}\right)x^3+O(x^4)\right)+e^{\frac{1}{x^3}-\frac{1}{3x}}x^{\frac{19}{3}+3U_{[3,7]}}\left(\_c_3+O(x)\right) \quad \textbf{(5.4)}$$

>

> $FormalSolution(eq, y(x), 'counterexample'='Eqs')$:

> $Eqs[1]$

$$\left(-3x^5+O(x^6)\right)y(x)+\left(3x^3+2x^2+x+1+2x^4+O(x^5)\right)\theta(y(x),x,1)+\left(3x+4x^5+O(x^6)\right)\theta(y(x),x,2)+\left(x^4-3x^7+O(x^8)\right)\theta(y(x),x,3)=0 \quad \textbf{(5.5)}$$

> $FormalSolution(Eqs[1], y(x))$

(5.6)

$$\left[\_c_1 + \frac{3\,\_c_1 x^5}{5} + O(x^6) + e^{\frac{1}{3x}} x^{2/3}\left(\_c_2 + \frac{35\,\_c_2 x}{27} + \frac{8947\,\_c_2 x^2}{1458} + \frac{5823683\,\_c_2 x^3}{118098} + O(x^4)\right) + \frac{e^{\frac{1}{x^3} - \frac{1}{3x}}\left(\_c_3 + O(x)\right)}{x^{8/3}}\right] \quad (5.6)$$

> *Eqs*[2]

$$\left(-5x^5 + O(x^6)\right) y(x) + \left(3x^3 + 2x^2 + x + 1 - 5x^4 + O(x^5)\right)\theta(y(x), x, 1) + \left(-3x^5 + 3x + O(x^6)\right)\theta(y(x), x, 2) + \left(-x^7 + x^4 + O(x^8)\right)\theta(y(x), x, 3) = 0 \quad (5.7)$$

>

> *FormalSolution*(*Eqs*[2], *y*(*x*))

$$\left[\_c_1 x^5 + \_c_1 + O(x^6) + e^{\frac{1}{3x}} x^{2/3}\left(\_c_2 + \frac{35\,\_c_2 x}{27} + \frac{8947\,\_c_2 x^2}{1458} + \frac{5884919\,\_c_2 x^3}{118098} + O(x^4)\right) + e^{\frac{1}{x^3} - \frac{1}{3x}} x^{10/3}\left(\_c_3 + O(x)\right)\right] \quad (5.8)$$

>

>

Laurent solution, one irregular solution with unknown exponent lambda and one solution with a truncated exponent of the irregular part

$$> eq := (1 + O(x))\, y(x) + \left(x + O(x^3)\right)\left(\frac{d}{dx} y(x)\right) + \left(RootOf(\_Z^3 + \_Z - 1, 'index' = 1)\, x^4 + O(x^7)\right)\left(\frac{d^2}{dx^2} y(x)\right) + \left(x^9 + O(x^{10})\right)\left(\frac{d^3}{dx^3} y(x)\right) = 0 :$$

> *FormalSolution*(*eq*, *y*(*x*));

$$\left[\frac{\_c_1 + O(x)}{x} + e^{\frac{\frac{RootOf(\_Z^3 + \_Z - 1, index = 1)^2}{2} + \frac{1}{2}}{x^2}} y_{reg}(x) + e^{\frac{RootOf(\_Z^3 + \_Z - 1, index = 1)}{4x^4}} y_1(x)\right] \quad (6.1)$$

If, when calling the FormalSolution command, the optional argument 'output' = 'literal' is used, then the regular parts of the solution are calculated to the maximum degree and, in addition, terms are added with coefficients depending on some of literals. In some cases it is possible to obtain the expression for λ which also depends on literals.

> *FormalSolution*(*eq*, *y*(*x*),'*output*'='*literal*')

$$\frac{-U_{[0,1]}\,\_c_1 x + \_c_1 + O(x^2)}{x} + e^{\frac{\frac{RootOf(\_Z^3 + \_Z - 1, index = 1)^2}{2} + \frac{1}{2}}{x^2}} x^{-\left(U_{[1,3]} + 1\right) RootOf(\_Z^3 + \_Z - 1, index = 1)^2 - RootOf(\_Z^3 + \_Z - 1, index = 1) - U_{[1,3]} + 2}\left(\_c_2 + O(x)\right) + e^{\frac{RootOf(\_Z^3 + \_Z - 1, index = 1)}{4x^4}} y_1(x) \quad (6.2)$$

> *FormalSolution*(*eq*, *y*(*x*),'*counterexample*'='*Eqs*') :

>

> $Eqs[1];$

$$\left(1+O(x^2)\right)y(x)+\left(x+O(x^4)\right)\left(\frac{d}{dx}y(x)\right)+\left(RootOf(\_Z^3+\_Z-1, index=1)\,x^4+O(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(x^9+O(x^{11})\right)\left(\frac{d^3}{dx^3}y(x)\right)=0 \quad (6.3)$$

> $FormalSolution(Eqs[1], y(x))$

$$\left[\frac{\_c_1+O(x^2)}{x}+e^{\frac{\frac{RootOf(\_Z^3+\_Z-1, index=1)^2}{2}+\frac{1}{2}}{x^2}}x^{-RootOf(\_Z^3+\_Z-1, index=1)^2-RootOf(\_Z^3+\_Z-1, index=1)+2}\left(\_c_2+O(x)\right)+e^{\frac{RootOf(\_Z^3+\_Z-1, index=1)}{4x^4}}y_1(x)\right] \quad (6.4)$$

> $Eqs[2]$

$$\left(1+4x+O(x^2)\right)y(x)+\left(x+4x^3+O(x^4)\right)\left(\frac{d}{dx}y(x)\right)+\left(RootOf(\_Z^3+\_Z-1, index=1)\,x^4+O(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(x^9+3x^{10}+O(x^{11})\right)\left(\frac{d^3}{dx^3}y(x)\right)=0 \quad (6.5)$$

>

> $FormalSolution(Eqs[2], y(x))$

$$\left[\frac{-4\_c_1x+\_c_1+O(x^2)}{x}+e^{\frac{\frac{RootOf(\_Z^3+\_Z-1, index=1)^2}{2}+\frac{1}{2}}{x^2}}x^{-5\,RootOf(\_Z^3+\_Z-1, index=1)^2-RootOf(\_Z^3+\_Z-1, index=1)-2}\left(\_c_2+O(x)\right)+e^{\frac{RootOf(\_Z^3+\_Z-1, index=1)}{4x^4}-\frac{RootOf(\_Z^3+\_Z-1, index=1)}{x^3}}y_1(x)\right] \quad (6.6)$$

One can see that **(6.4)** and **(6.6)** are prolongations of **(6.1)**, they differ in the Laurent solution. The exponents λ of the second solutions are also different. In some cases we obtain a prolongation for exponents of the irregular part.

>

>

The RootOf in a solution

> $eq := \left(x^5+x^6+O(x^7)\right)\left(\frac{d^3}{dx^3}y(x)\right)+\left(-3x^3-x^4+O(x^5)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(1+x+O(x^2)\right)y(x)=0:$

> $FormalSolution(eq, y(x));$

(7.1)

$$\left[e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=1)}{\sqrt{x}}} x^{29/36} \left(\_c_1 + \frac{191\,RootOf(3\_Z^2-1,\,index=1)\,\_c_1\sqrt{x}}{432} - \frac{82679\,\_c_1\,x}{1119744} + O(x^{3/2}) \right) + e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=2)}{\sqrt{x}}} x^{29/36} \left(\_c_2 + \frac{191\,RootOf(3\_Z^2-1,\,index=2)\,\_c_2\sqrt{x}}{432} - \frac{82679\,\_c_2\,x}{1119744} + O(x^{3/2}) \right) + e^{-\frac{3}{x}} x^{17/9} \left(\_c_3 + O(x) \right) \right] \quad (7.1)$$

> *FormalSolution*(*eq*, *y*(*x*),'*output*'='*literal*')

$$e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=1)}{\sqrt{x}}} x^{29/36} \left(\_c_1 + \frac{191\,RootOf(3\_Z^2-1,\,index=1)\,\_c_1\sqrt{x}}{432} - \frac{82679\,\_c_1\,x}{1119744} + 9\,RootOf(3\_Z^2-1,\,index=1) \left(-\frac{170149537}{13060694016}\,\_c_1 + \frac{1}{27}\,\_c_1\,U_{[0,2]} + \frac{1}{81}\,\_c_1\,U_{[2,5]} \right) x^{3/2} + O(x^2) \right) + e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=2)}{\sqrt{x}}} x^{29/36} \left(\_c_2 + \frac{191\,RootOf(3\_Z^2-1,\,index=2)\,\_c_2\sqrt{x}}{432} - \frac{82679\,\_c_2\,x}{1119744} + 9\,RootOf(3\_Z^2-1,\,index=2) \left(-\frac{170149537}{13060694016}\,\_c_2 + \frac{1}{27}\,\_c_2\,U_{[0,2]} + \frac{1}{81}\,\_c_2\,U_{[2,5]} \right) x^{3/2} + O(x^2) \right) + e^{-\frac{3}{x}} x^{17/9} \left(\_c_3 + \left(\frac{358}{243}\,\_c_3 - \_c_3\,U_{[2,5]} - 3\,\_c_3\,U_{[3,7]} \right) x + O(x^2) \right) \quad (7.2)$$

> *FormalSolution*(*eq*, *y*(*x*),'*counterexample*'='*Eqs*', *top* = infinity) :

>

> *Eqs*[1];

$$\left(1 + x + 5x^2 + O(x^3)\right) y(x) + \left(-x^4 - 3x^3 - 4x^5 + O(x^6)\right) \left(\frac{d^2}{dx^2}\,y(x)\right) + \left(x^6 + x^5 + 2x^7 + O(x^8)\right) \left(\frac{d^3}{dx^3}\,y(x)\right) = 0 \quad (7.3)$$

> *FormalSolution*(*Eqs*[1], *y*(*x*))

$$\left[e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=1)}{\sqrt{x}}} x^{29/36} \left(\_c_1 + \frac{191\,RootOf(3\_Z^2-1,\,index=1)\,\_c_1\sqrt{x}}{432} - \frac{82679\,\_c_1\,x}{1119744} + \frac{1603524959\,RootOf(3\_Z^2-1,\,index=1)\,\_c_1\,x^{3/2}}{1451188224} - \frac{4454530543343\,\_c_1\,x^2}{7522959753216} + O(x^{5/2}) \right) + e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=2)}{\sqrt{x}}} x^{29/36} \left(\_c_2 + \frac{191\,RootOf(3\_Z^2-1,\,index=2)\,\_c_2\sqrt{x}}{432} - \frac{82679\,\_c_2\,x}{1119744} + \frac{1603524959\,RootOf(3\_Z^2-1,\,index=2)\,\_c_2\,x^{3/2}}{1451188224} - \frac{4454530543343\,\_c_2\,x^2}{7522959753216} + O(x^{5/2}) \right) + e^{-\frac{3}{x}} x^{17/9} \left(\_c_3 - \frac{128\,\_c_3\,x}{243} + O(x^2) \right) \right] \quad (7.4)$$

> $Eqs[2]$

$$\left(1+x-2x^2+O(x^3)\right)y(x)+\left(-x^4-3x^3-4x^5+O(x^6)\right)\left(\frac{d^2}{dx^2}y(x)\right)+\left(x^6+x^5+3x^7+O(x^8)\right)\left(\frac{d^3}{dx^3}y(x)\right)=0 \quad (7.5)$$

> $FormalSolution(Eqs[2], y(x))$

$$\left[e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=1)}{\sqrt{x}}}x^{29/36}\left(\_c_1+\frac{191\,RootOf(3\_Z^2-1,\,index=1)\,\_c_1\sqrt{x}}{432}-\frac{82679\,\_c_1 x}{1119744}-\frac{1782580897\,RootOf(3\_Z^2-1,\,index=1)\,\_c_1 x^{3/2}}{1451188224}+\frac{4869837525265\,\_c_1 x^2}{7522959753216}+O(x^{5/2})\right)+e^{-\frac{2\,RootOf(3\_Z^2-1,\,index=2)}{\sqrt{x}}}x^{29/36}\left(\_c_2+\frac{191\,RootOf(3\_Z^2-1,\,index=2)\,\_c_2\sqrt{x}}{432}-\frac{82679\,\_c_2 x}{1119744}-\frac{1782580897\,RootOf(3\_Z^2-1,\,index=2)\,\_c_2 x^{3/2}}{1451188224}+\frac{4869837525265\,\_c_2 x^2}{7522959753216}+O(x^{5/2})\right)+e^{-\frac{3}{x}}x^{17/9}\left(\_c_3-\frac{857\,\_c_3 x}{243}+O(x^2)\right)\right] \quad (7.6)$$

>

>

One more example with the RootOf in a solution

> $eq := \left(-x^2+O(x^4)\right)\theta(y(x),x,5)+O(x^3)\,\theta(y(x),x,4)+\left(x+O(x^3)\right)\theta(y(x),x,3)+O(x^3)\,\theta(y(x),x,2)+\left(x+O(x^3)\right)\theta(y(x),x,1)+\left(1+O(x)\right)y(x):$

> $FormalSolution(eq, y(x));$

$$\left[e^{-\frac{3\,RootOf(\_Z^2-\_Z+1,\,index=1)}{x^{1/3}}}\left(\_c_1+\left(\frac{20}{9}-\frac{20\,RootOf(\_Z^2-\_Z+1,\,index=1)}{9}\right)\_c_1 x^{1/3}+O(x^{2/3})\right)+e^{-\frac{3\,RootOf(\_Z^2-\_Z+1,\,index=2)}{x^{1/3}}}\left(\_c_2+\left(\frac{20}{9}-\frac{20\,RootOf(\_Z^2-\_Z+1,\,index=2)}{9}\right)\_c_2 x^{1/3}+O(x^{2/3})\right)+e^{\frac{3}{x^{1/3}}}\left(\_c_3-\frac{20\,\_c_3 x^{1/3}}{9}+O(x^{2/3})\right)+e^{\frac{2}{\sqrt{x}}}x^{9/4}\left(\_c_4+\frac{161\,\_c_4\sqrt{x}}{16}+O(x)\right)+e^{-\frac{2}{\sqrt{x}}}x^{9/4}\left(\_c_5-\frac{161\,\_c_5\sqrt{x}}{16}+O(x)\right)\right] \quad (8.1)$$

> $FormalSolution(eq, y(x), 'output'='literal')$

$$e^{\frac{3}{x^{1/3}}}\left(\_c_1-\frac{20\,\_c_1 x^{1/3}}{9}+\left(-\frac{125}{162}\_c_1-\frac{1}{2}\_c_1 U_{[0,1]}\right)x^{2/3}+O(x)\right)+e^{-\frac{3\,RootOf(\_Z^2-\_Z+1,\,index=1)}{x^{1/3}}}\left(\_c_2+\left(\frac{20}{9}-\frac{20\,RootOf(\_Z^2-\_Z+1,\,index=1)}{9}\right)\_c_2 x^{1/3}\right. \quad (8.2)$$

$$- RootOf(\_Z^2 - \_Z + 1, index = 1) \left(-\frac{125}{162}\_c_2 - \frac{1}{2}\_c_2 U_{[0,1]} \right) x^{2/3} + O(x) \Big)$$

$$+ e^{-\frac{3 RootOf(\_Z^2 - \_Z + 1, index=2)}{x^{1/3}}} \Big(\_c_3 + \left(\frac{20}{9} - \frac{20 RootOf(\_Z^2 - \_Z + 1, index = 2)}{9} \right) \_c_3 x^{1/3}$$

$$- RootOf(\_Z^2 - \_Z + 1, index = 2) \left(-\frac{125}{162}\_c_3 - \frac{1}{2}\_c_3 U_{[0,1]} \right) x^{2/3} + O(x) \Big)$$

$$+ e^{-\frac{2}{\sqrt{x}}} x^{9/4} \left(\_c_4 - \frac{161 \_c_4 \sqrt{x}}{16} + \left(\frac{42825}{512}\_c_4 + \frac{1}{2}\_c_4 U_{[4,3]} + \frac{1}{2}\_c_4 U_{[0,1]} \right) x + O(x^{3/2}) \right)$$

$$+ e^{\frac{2}{\sqrt{x}}} x^{9/4} \left(\_c_5 + \frac{161 \_c_5 \sqrt{x}}{16} + \left(\frac{42825}{512}\_c_5 + \frac{1}{2}\_c_5 U_{[4,3]} + \frac{1}{2}\_c_5 U_{[0,1]} \right) x + O(x^{3/2}) \right)$$

```
> FormalSolution(eq, y(x),'counterexample'='Eqs') :
>
> Eqs[1];
```

$$(1 - x + O(x^2))\, y(x) + (x + O(x^3))\, \theta(y(x), x, 1) + O(x^3)\, \theta(y(x), x, 2) + (x + O(x^3))\, \theta(y(x), x, 3) + (3x^3 + O(x^4))\, \theta(y(x), x, 4) + (-x^2 + O(x^4))\, \theta(y(x), x, 5) = 0 \quad \textbf{(8.3)}$$

```
> FormalSolution(Eqs[1], y(x))
```

$$\Big[e^{-\frac{3 RootOf(\_Z^2 - \_Z + 1, index=1)}{x^{1/3}}} \Big(\_c_1 + \left(\frac{20}{9} - \frac{20 RootOf(\_Z^2 - \_Z + 1, index = 1)}{9} \right) \_c_1 x^{1/3} + \frac{22 RootOf(\_Z^2 - \_Z + 1, index = 1)\, \_c_1 x^{2/3}}{81} - \frac{877 \_c_1 x}{2187} + \left(-\frac{94231 RootOf(\_Z^2 - \_Z + 1, index = 1)}{78732} + \frac{94231}{78732} \right) \_c_1 x^{4/3} + O(x^{5/3}) \Big)$$

$$+ e^{-\frac{3 RootOf(\_Z^2 - \_Z + 1, index=2)}{x^{1/3}}} \Big(\_c_2 + \left(\frac{20}{9} - \frac{20 RootOf(\_Z^2 - \_Z + 1, index = 2)}{9} \right) \_c_2 x^{1/3} + \frac{22 RootOf(\_Z^2 - \_Z + 1, index = 2)\, \_c_2 x^{2/3}}{81} - \frac{877 \_c_2 x}{2187} + \left(-\frac{94231 RootOf(\_Z^2 - \_Z + 1, index = 2)}{78732} + \frac{94231}{78732} \right) \_c_2 x^{4/3} + O(x^{5/3}) \Big) + e^{\frac{3}{x^{1/3}}} \Big(\_c_3 - \frac{20 \_c_3 x^{1/3}}{9} - \frac{22 \_c_3 x^{2/3}}{81} - \frac{877 \_c_3 x}{2187} - \frac{94231 \_c_3 x^{4/3}}{78732} + O(x^{5/3}) \Big) + e^{\frac{2}{\sqrt{x}}} x^{9/4} \Big(\_c_4 + \frac{161 \_c_4 \sqrt{x}}{16} + \frac{43337 \_c_4 x}{512} + O(x^{3/2}) \Big) + e^{-\frac{2}{\sqrt{x}}} x^{9/4} \Big(\_c_5 - \frac{161 \_c_5 \sqrt{x}}{16} + \frac{43337 \_c_5 x}{512} + O(x^{3/2}) \Big) \Big] \quad \textbf{(8.4)}$$

```
> Eqs[2]
```

$$(1 + x + O(x^2))\, y(x) + (x + O(x^3))\, \theta(y(x), x, 1) + O(x^3)\, \theta(y(x), x, 2) + (x + O(x^3))\, \theta(y(x), x, 3) + (-2x^3 + O(x^4))\, \theta(y(x), x, 4) + (-x^2 + O(x^4))\, \theta(y(x), x, 5) = 0 \quad \textbf{(8.5)}$$

```
> FormalSolution(Eqs[2], y(x))
```

$$\left[e^{-\frac{3\,RootOf(\_Z^2 - \_Z + 1,\, index=1)}{x^{1/3}}} \left(\_c_1 + \left(\frac{20}{9} - \frac{20\,RootOf(\_Z^2 - \_Z + 1,\, index=1)}{9} \right) \_c_1\, x^{1/3} + \frac{103\,RootOf(\_Z^2 - \_Z + 1,\, index=1)\,\_c_1\, x^{2/3}}{81} + \frac{1067\,\_c_1\, x}{2187} + \left(\frac{37010\,RootOf(\_Z^2 - \_Z + 1,\, index=1)}{19683} - \frac{37010}{19683} \right) \_c_1\, x^{4/3} + O(x^{5/3}) \right) + e^{-\frac{3\,RootOf(\_Z^2 - \_Z + 1,\, index=2)}{x^{1/3}}} \left(\_c_2 + \left(\frac{20}{9} - \frac{20\,RootOf(\_Z^2 - \_Z + 1,\, index=2)}{9} \right) \_c_2\, x^{1/3} + \frac{103\,RootOf(\_Z^2 - \_Z + 1,\, index=2)\,\_c_2\, x^{2/3}}{81} + \frac{1067\,\_c_2\, x}{2187} + \left(\frac{37010\,RootOf(\_Z^2 - \_Z + 1,\, index=2)}{19683} - \frac{37010}{19683} \right) \_c_2\, x^{4/3} + O(x^{5/3}) \right) + e^{\frac{3}{x^{1/3}}} \left(\_c_3 - \frac{20\,\_c_3\, x^{1/3}}{9} - \frac{103\,\_c_3\, x^{2/3}}{81} + \frac{1067\,\_c_3\, x}{2187} + \frac{37010\,\_c_3\, x^{4/3}}{19683} + O(x^{5/3}) \right) + e^{\frac{2}{\sqrt{x}}}\, x^{9/4} \left(\_c_4 + \frac{161\,\_c_4 \sqrt{x}}{16} + \frac{42569\,\_c_4\, x}{512} + O(x^{3/2}) \right) + e^{-\frac{2}{\sqrt{x}}}\, x^{9/4} \left(\_c_5 - \frac{161\,\_c_5 \sqrt{x}}{16} + \frac{42569\,\_c_5\, x}{512} + O(x^{3/2}) \right) \right] \tag{8.6}$$

>

>

The RootOf in an equation

The irrational algebraic numbers involved in the equation must be represented using RootOf(expr, index=i) where expr is an irreducible polynomial in _Z, the i-th root of which is the necessary algebraic number.

$$> eq := (1 + O(x))\, y(x) + (x + O(x^3)) \left(\frac{d}{dx} y(x) \right) + \left(RootOf(\_Z^3 + \_Z - 1, 'index' = 1)\, x^4 + O(x^7) \right) \left(\frac{d^2}{dx^2} y(x) \right) + (x^9 + O(x^{10})) \left(\frac{d^3}{dx^3} y(x) \right) = 0 :$$

$$> FormalSolution(eq, y(x));$$

$$\left[\frac{\_c_1 + O(x)}{x} + e^{\frac{\frac{RootOf(\_Z^3 + \_Z - 1,\, index=1)^2}{2} + \frac{1}{2}}{x^2}}\, y_{reg}(x) + e^{\frac{RootOf(\_Z^3 + \_Z - 1,\, index=1)}{4 x^4}}\, y_1(x) \right] \tag{9.1}$$

$$> FormalSolution(eq, y(x), 'output'='literal')$$

$$\frac{-U_{[0,1]}\, \_c_1\, x + \_c_1 + O(x^2)}{x} + e^{\frac{\frac{RootOf(\_Z^3 + \_Z - 1,\, index=1)^2}{2} + \frac{1}{2}}{x^2}}\, x^{-(U_{[1,3]} + 1)\, RootOf(\_Z^3 + \_Z - 1,\, index=1)^2 - RootOf(\_Z^3 + \_Z - 1,\, index=1) - U_{[1,3]} + 2} \left(\_c_2 + O(x) \right) \tag{9.2}$$

$$+ e^{\frac{RootOf(\_Z^3+\_Z-1, index=1)}{4x^4}} y_1(x)$$

> *FormalSolution*(*eq*, *y*(*x*),'*counterexample*'='*Eqs*') :

>

> *Eqs*[1];

$$(1+O(x^2))\,y(x)+(x+O(x^4))\left(\frac{d}{dx}\,y(x)\right)+\left(RootOf(\_Z^3+\_Z-1, index=1)\,x^4+O(x^7)\right)\left(\frac{d^2}{dx^2}\,y(x)\right)+(x^9+O(x^{11}))\left(\frac{d^3}{dx^3}\,y(x)\right)=0 \quad (9.3)$$

> *FormalSolution*(*Eqs*[1], *y*(*x*))

$$\left[\frac{\_c_1+O(x^2)}{x}, e^{\frac{\frac{RootOf(\_Z^3+\_Z-1, index=1)^2}{2}+\frac{1}{2}}{x^2}} x^{-RootOf(\_Z^3+\_Z-1, index=1)^2-RootOf(\_Z^3+\_Z-1, index=1)+2}\left(\_c_2+O(x)\right)+e^{\frac{RootOf(\_Z^3+\_Z-1, index=1)}{4x^4}} y_1(x)\right] \quad (9.4)$$

> *Eqs*[2]

$$(1+4x+O(x^2))\,y(x)+(x+3x^3+O(x^4))\left(\frac{d}{dx}\,y(x)\right)+\left(RootOf(\_Z^3+\_Z-1, index=1)\,x^4+O(x^7)\right)\left(\frac{d^2}{dx^2}\,y(x)\right)+(x^9+x^{10}+O(x^{11}))\left(\frac{d^3}{dx^3}\,y(x)\right)=0 \quad (9.5)$$

> *FormalSolution*(*Eqs*[2], *y*(*x*))

$$\left[\frac{-4\,\_c_1\,x+\_c_1+O(x^2)}{x}, e^{\frac{\frac{RootOf(\_Z^3+\_Z-1, index=1)^2}{2}+\frac{1}{2}}{x^2}} x^{-4\,RootOf(\_Z^3+\_Z-1, index=1)^2-RootOf(\_Z^3+\_Z-1, index=1)-1}\left(\_c_2+O(x)\right)+e^{\frac{RootOf(\_Z^3+\_Z-1, index=1)}{4x^4}-\frac{RootOf(\_Z^3+\_Z-1, index=1)}{3x^3}} y_1(x)\right] \quad (9.6)$$

>

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