

[time: by Maple 2015, Ubuntu 8.04.4 LTS, AMD Athlon(tm) 64 Processor 3700+, 3GB RAM

> restart;

> read "lrshypergeomsolscasc2015.mpl";

## Example 1

$$\text{> } AI := \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)x} & \frac{x^2+x+1}{x} \end{bmatrix} :$$

The procedure returns a resolving sequence and a resolving matrix for  $Y(x+1)=A1.Y(x)$  starting from  $y[1](x)$ .

> *Eqs, B := LRS:-Resolving(A1, x, y, indicator = 1) :*

*Eqs[1]=0; Eqs[2]=0;*

$$\begin{aligned} & (-x^6 - x^5 + 4x^4 + 3x^3 - 3x^2 - 2x) y(x) + (2x^6 + 6x^5 + 2x^4 - 6x^3 - 6x^2 - 4x) y(x \\ & + 1) + (-x^6 - 6x^5 - 11x^4 - 2x^3 + 8x^2 + 8x + 4) y(x + 2) + (x^5 + 4x^4 + x^3 \\ & - 6x^2) y(x + 3) = 0 \\ & \qquad \qquad \qquad -y(x) x + y(x + 1) = 0 \end{aligned} \tag{1.1}$$

The procedure returns an equivalent single equation and a matrix B(x) by the cyclic vector method

> *LRS:-CyclicVector(A1, x, y)[1, 1]=0;*

$$\begin{aligned} & (216x^{16} + 2736x^{15} + 13848x^{14} + 13896x^{13} - 225046x^{12} - 1279404x^{11} - 2590626x^{10} \\ & + 148424x^9 + 9058465x^8 + 12241281x^7 - 182690x^6 - 11062671x^5 - 6480547x^4 \\ & - 119042x^3 + 461160x^2) y(x) + (-432x^{16} - 6552x^{15} - 41520x^{14} - 93696x^{13} \\ & + 464852x^{12} + 4572454x^{11} + 17338588x^{10} + 36072472x^9 + 38235226x^8 \\ & + 3556419x^7 - 41619459x^6 - 50072690x^5 - 27167229x^4 - 9544887x^3 \\ & - 3290586x^2 - 673640x) y(x + 1) + (216x^{16} + 4248x^{15} + 34440x^{14} + 122616x^{13} \\ & - 152038x^{12} - 3935334x^{11} - 21138966x^{10} - 63858516x^9 - 116526301x^8 \\ & - 109403774x^7 + 17271111x^6 + 187694014x^5 + 233803892x^4 + 126789448x^3 \\ & + 13309796x^2 - 16616792x - 5806240) y(x + 2) + (-432x^{15} - 6768x^{14} \\ & - 43104x^{13} - 94128x^{12} + 557732x^{11} + 5759796x^{10} + 24865732x^9 + 62152912x^8 \\ & + 85434160x^7 + 28885218x^6 - 104044129x^5 - 181250808x^4 - 117491893x^3 \\ & - 15097092x^2 + 17694964x + 6847280) y(x + 3) + (216x^{14} + 2736x^{13} \\ & + 13128x^{12} + 8544x^{11} - 291958x^{10} - 2057116x^9 - 7030722x^8 - 12767046x^7 \\ & - 8438475x^6 + 11305591x^5 + 27629889x^4 + 20515827x^3 + 3232762x^2 - 3139056x \\ & - 1314720) y(x + 4) = 0 \end{aligned} \tag{1.2}$$

Set to see details of the process and the CPU time of some steps:

```
> infolevel[LRS] := 3 :
```

The procedure returns a basis of hypergeom solutions space for  $Y(x+1)=A1.Y(x)$ .

One-dimension hypergeometric solutions space is found

```
> Res := LRS:-HypergeometricSolution(A1, x);  
evalb(nops(Res) = 1 and {seq(simplify(convert(eval(Res[i], x = x + 1) - A1.Res[i],  
set)) [ ], i = 1 .. nops(Res))) = {0}});
```

Resolving: resolving for [1 0 0 0] : 0.

Resolving: 3 order resolving equation is constructed: 0.20e-1

Resolving: A is constructed: 0.14e-1

Resolving: resolving for [1] : 0.

Resolving: 1 order resolving equation is constructed: 0.6e-2

HypergeometricSolution: 1 dimension hypergeometric solution space is found: .170

ByRationalSolution: there are no rational solutions: 0.57e-1

HypergeometricSolution: 1 dimension hypergeometric solution space is found: 0.13e-1

ByRationalSolution: 1 dimension rational solution space are found: 0.37e-1

HypergeometricSolution: all time: .334

$$Res := \begin{bmatrix} 0 \\ -\Gamma(x) \\ 0 \\ \Gamma(x) \end{bmatrix}$$

*true*

(1.3)

The same result by the cyclic vector method. It takes more time:

```
> Res := LRS:-HypergeometricSolution(A1, x, by_cyclic_vector = true) ;  
evalb(nops(Res) = 1 and {seq(simplify(convert(eval(Res[i], x = x + 1) - A1.Res[i],  
set)) [ ], i = 1 .. nops(Res))) = {0}});
```

CyclicVector: check if [4 2 -5 -9] is a cyclic vector

CyclicVector: 4 order resolving equation is constructed: 0.80e-1

HypergeometricSolution: 1 dimension hypergeometric solution space is found: .354

ByLinearSolve: LinearSolve: 0.7e-2

HypergeometricSolution: all time: .474

*true*

(1.4)

## Example 2

$$A2 := \begin{bmatrix} \frac{x^3 + 4x^2 + 4x - 2}{(x+4)(x+2)(x+1)} & \frac{x^2 + 3x + 1}{(x+2)(x+1)} & \frac{x+1}{x+4} & \frac{2x+4}{x+4} \\ -\frac{x^3 + 4x^2 + 4x - 2}{(x+4)(x+2)(x+1)} & \frac{1}{(x+2)(x+1)} & -\frac{x+1}{x+4} & -\frac{x}{x+4} \\ -\frac{x(2x^2 + 8x + 9)}{(x+4)(x+2)(x+1)} & -\frac{x^2 + 3x + 1}{(x+2)(x+1)} & -\frac{2x+2}{x+4} & -\frac{2x}{x+4} \\ \frac{x+1}{x+4} & 0 & \frac{x+1}{x+4} & \frac{x}{x+4} \end{bmatrix} :$$

> infolevel[LRS] := 3 :

Four-dimension hypergeometric solutions space is found, starting from y[4](x)

> Res := LRS:-HypergeometricSolution(A2, x);  
 evalb(nops(Res) = 4 and {seq(simplify(convert(eval(Res[i], x=x+1) - A2.Res[i], set)))[ ], i=1..nops(Res))} = {0});

Resolving: resolving for [0 0 0 1] : 0.3e-2

Resolving: 2 order resolving equation is constructed: 0.10e-1

Resolving: A is constructed: 0.14e-1

Resolving: resolving for [1 0] : 0.

Resolving: 2 order resolving equation is constructed: 0.6e-2

HypergeometricSolution: 2 dimension hypergeometric solution space is found: 0.40e-1

ByRationalSolution: 1 dimension rational solution space are found: .100

ByRationalSolution: 1 dimension rational solution space are found: 0.50e-1

HypergeometricSolution: 2 dimension hypergeometric solution space is found: 0.27e-1

ByRationalSolution: 1 dimension rational solution space are found: 0.70e-1

ByRationalSolution: 1 dimension rational solution space are found: 0.57e-1

HypergeometricSolution: all time: .450

$$Res := \left[ \left[ \begin{array}{c} \frac{1}{(x+3)(x+2)} \\ -\frac{1}{(x+3)(x+2)} \\ -\frac{x-1}{(x+1)(x+2)(x+3)} \\ \frac{1}{(x+3)(x+2)} \end{array} \right], \left[ \begin{array}{c} \frac{(-1)^x(2x+5)}{(x+2)(x+3)} \\ -\frac{(-1)^x(2x+5)}{(x+2)(x+3)} \\ -\frac{(6x^2+23x+19)(-1)^x}{(x+1)(x+3)(x+2)} \\ \frac{(-1)^x(2x+5)}{(x+2)(x+3)} \end{array} \right] \right],$$

$$\begin{bmatrix} -\frac{(x+2)(-1)^x}{\Gamma(x+2)} \\ \frac{(-1)^x}{\Gamma(x+2)} \\ \frac{(x+2)(-1)^x}{\Gamma(x+2)} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{x}{\Gamma(x+2)} \\ -\frac{1}{\Gamma(x+2)} \\ \frac{x}{\Gamma(x+2)} \\ 0 \end{bmatrix}$$

true

(2.1)

The same result starting from  $y[1](x)$ . It takes more time:

> Res := LRS:-HypergeometricSolution(A2, x, select\_indicator = 1) :  
 evalb(nops(Res) = 4 and {seq(simplify(convert(eval(Res[i], x = x + 1) - A2.Res[i], set)) [ ], i = 1 .. nops(Res))} = {0});

Resolving: resolving for [1 0 0 0] : 0.4e-2

Resolving: 4 order resolving equation is constructed: 0.36e-1

HypergeometricSolution: 4 dimension hypergeometric solution space is found: .364

ByLinearSolve: LinearSolve: 0.10e-1

HypergeometricSolution: all time: .560

true

(2.2)

Off to unsee details of the process:

> infolevel[LRS] := 1 :

The resolving sequence starting from  $y[1](x)$  has two equations:

> L, B := LRS:-Resolving(A2, x, y, indicator = 1) :

L[1] = 0;

$(x^9 + 30x^8 + 393x^7 + 2939x^6 + 13755x^5 + 41483x^4 + 79808x^3 + 93023x^2 + 58092x$

$+ 14028) y(x) + (x^8 + 30x^7 + 383x^6 + 2727x^5 + 11919x^4 + 33035x^3 + 57308x^2$

$+ 57507x + 25746) y(x+1) + (-x^{11} - 33x^{10} - 486x^9 - 4205x^8 - 23663x^7$

$- 90474x^6 - 238043x^5 - 425849x^4 - 496578x^3 - 342247x^2 - 107775x - 2310)$

$y(x+2) + (x^{10} + 29x^9 + 367x^8 + 2632x^7 + 11524x^6 + 30188x^5 + 38397x^4$

$- 14173x^3 - 124139x^2 - 165186x - 76440) y(x+3) + (x^{11} + 36x^{10} + 572x^9$

$+ 5279x^8 + 31272x^7 + 123672x^6 + 327545x^5 + 561714x^4 + 563308x^3 + 227419x^2$

$- 81018x - 76440) y(x+4) = 0$

(2.3)

The resolving sequence starting from  $y[4](x)$  has one equation:

> L, B := LRS:-Resolving(A2, x, y, indicator = 4) :

L[1] = 0;

L[2] = 0;

$$(-x - 2) y(x) + y(x + 1) + (x + 5) y(x + 2) = 0$$

$$-y(x) + (x^2 + 5x + 6) y(x + 2) = 0$$

(2.4)

## Example 16 × 16

### input A16

The matrix A16 is sparse. Here is its structure and one nonzero element:

```
> tmp := map(el → `if`(el ≠ 0, "", 0), A16) :
tmp[1..8, 1..8], tmp[1..8, 9..16];
tmp[9..16, 1..8], tmp[9..16, 9..16];
A16[4, 2];
```

$$\begin{bmatrix}
 0 & "" & 0 & 0 & 0 & "" & 0 & 0 \\
 0 & 0 & "" & 0 & 0 & 0 & "" & 0 \\
 0 & 0 & 0 & "" & 0 & 0 & 0 & "" \\
 "" & "" & "" & "" & "" & "" & "" & "" \\
 0 & "" & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & "" & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & "" & 0 & 0 & 0 & 0 \\
 "" & "" & "" & "" & 0 & 0 & 0 & 0
 \end{bmatrix},
 \begin{bmatrix}
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & "" & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & "" & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & "" & 0 & 0 & 0 & 0 \\
 "" & "" & "" & "" & 0 & 0 & 0 & 0
 \end{bmatrix}$$
  

$$\begin{bmatrix}
 0 & "" & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & "" & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & "" & 0 & 0 & 0 & 0 \\
 "" & "" & "" & "" & 0 & 0 & 0 & 0 \\
 0 & "" & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & "" & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & "" & 0 & 0 & 0 & 0 \\
 "" & "" & "" & "" & 0 & 0 & 0 & 0
 \end{bmatrix},
 \begin{bmatrix}
 0 & 0 & 0 & 0 & 0 & "" & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & "" & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & "" \\
 0 & 0 & 0 & 0 & "" & "" & "" & "" \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

$$\frac{3}{4} \left( (2025x^4 + 10800x^3 + 19971x^2 + 15240x + 4228)^2 x (15x + 34) \right) / \left( (19 + 15x) (9x^2 + 27x + 22) (15x + 4) (3x + 8) (3x + 5) (9x^2 + 9x + 4) (x + 2)^2 \right) \quad (3.1)$$

```
> infolevel[LRS] := 3 :
```

Two-dimensional solutions space is found by the resolving method:

```
> Res := LRS:-HypergeometricSolution(A16, x);
evalb(nops(Res) = 2 and {seq(simplify(convert(eval(Res[i], x = x + 1) - A16.Res[i],
set)) [ ], i = 1..nops(Res))) = {0}});
```

```
Resolving: resolving for [0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0] :
0.3e-2
```

```
Resolving: 15 order resolving equation is constructed: 30.910
```

```
Resolving: A is constructed: 1.950
```

```
Resolving: resolving for [1] : 0.
```

```
Resolving: 1 order resolving equation is constructed: 0.3e-2
```

HypergeometricSolution: 1 dimension hypergeometric solution space is found: 294.524  
 ByRationalSolution: 2 dimension rational solution space are found: .950  
 HypergeometricSolution: 1 dimension hypergeometric solution space is found: 0.13e-1  
 HypergeometricSolution: all time: 328.416

$$Res := \left[ \left[ \begin{array}{l} 1 .. 16 \text{ Vector}_{column} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array} \right], \left[ \begin{array}{l} 1 .. 16 \text{ Vector}_{column} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array} \right] \right]$$

true

(3.2)

The same result by the cyclic vector metod:

> Res := LRS:-HypergeometricSolution(A16, x, by\_cyclic\_vector = true);  
 evalb(nops(Res) = 2 and {seq(simplify(convert(eval(Res[i], x=x+1) - A16.Res[i], set)) [ ], i=1..nops(Res))) = {0}});

CyclicVector: check if [4 2 -5 -9 1 -7 -5 -6 6 -3 -6 -9 7 9 0 8] is a cyclic vector

CyclicVector: 16 order resolving equation is constructed:  
 121.633

HypergeometricSolution: 2 dimension hypergeometric solution space is found: 1413.984

ByLinearSolve: LinearSolve: 61.714

HypergeometricSolution: all time: 1597.607

$$Res := \left[ \left[ \begin{array}{l} 1 .. 16 \text{ Vector}_{column} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array} \right], \left[ \begin{array}{l} 1 .. 16 \text{ Vector}_{column} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array} \right] \right]$$

true

(3.3)

All solutions are hypergeometric:

> convert(Res[1], list);  
 convert(Res[2], list);

$$\left[ -\frac{1}{3} \frac{45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (6583545 x^3 + 8373357 x^2 + 4697592 x + 780272)}{9 x^2 + 9 x + 4}, \right. \\ \left. -\frac{2280 \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x (3 x^2 + 5 x + 2)}{9 x^2 + 9 x + 4}, \right. \\ \left. \frac{1520 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (3 x + 2)}{(9 x^2 + 9 x + 4) x}, \right. \\ \left. \frac{2280 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (3 x + 2)}{(9 x^2 + 9 x + 4) x}, \right]$$

$$\begin{aligned}
& \frac{380 (270 x^4 + 1377 x^3 + 1728 x^2 + 413 x - 48) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x x}{135 x^3 + 306 x^2 + 231 x + 76}, \\
& - \frac{1}{3} \frac{1}{135 x^3 + 306 x^2 + 231 x + 76} \left( (97983675 x^4 + 248122710 x^3 + 226927683 x^2 \right. \\
& \left. + 100233288 x + 14861648) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x \right), \\
& - \frac{760 (135 x^3 + 486 x^2 + 585 x + 214) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x}{135 x^3 + 306 x^2 + 231 x + 76}, \\
& - \frac{380 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (675 x^3 + 2070 x^2 + 1869 x + 562)}{135 x^4 + 441 x^3 + 537 x^2 + 307 x + 76}, \\
& - \frac{380}{3} \frac{1}{135 x^3 + 306 x^2 + 231 x + 76} \left( (2025 x^5 + 5265 x^4 + 3465 x^3 - 1197 x^2 \right. \\
& \left. - 1718 x - 360) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x x \right), \\
& - \frac{760 \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x (135 x^5 + 261 x^4 - 21 x^3 - 321 x^2 - 214 x - 40)}{135 x^3 + 306 x^2 + 231 x + 76}, \\
& - \frac{1}{135 x^3 + 306 x^2 + 231 x + 76} \left( (32661225 x^4 + 82570770 x^3 + 75264081 x^2 \right. \\
& \left. + 33075176 x + 4857616) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x \right), \\
& \frac{1140 (90 x^3 + 309 x^2 + 187 x + 44) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x}{135 x^3 + 306 x^2 + 231 x + 76}, \\
& \frac{1}{135 x^3 + 306 x^2 + 231 x + 76} \left( 760 \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x x (135 x^6 + 486 x^5 \right. \\
& \left. + 450 x^4 - 212 x^3 - 545 x^2 - 274 x - 40) \right), \\
& - \frac{1520}{3} \frac{1}{135 x^3 + 306 x^2 + 231 x + 76} \left( (135 x^5 + 621 x^4 + 1071 x^3 + 859 x^2 \right.
\end{aligned}$$

$$+ 314x + 40) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x),$$

$$- \frac{1}{135x^3 + 306x^2 + 231x + 76} \left( 760 (135x^5 + 621x^4 + 1071x^3 + 859x^2 + 314x + 40) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x \right),$$

$$- \frac{1}{135x^3 + 306x^2 + 231x + 76} \left( (32917725x^4 + 83425770x^3 + 76277541x^2 + 33544856x + 4936656) \Gamma\left(x + \frac{2}{3}\right) \Gamma\left(x + \frac{4}{15}\right) 45^x \right) \Big]$$

$$\left[ -\frac{243}{608} 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (15x + 4), 0, 0, 0, 0, -\frac{243}{608} 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (15x + 4), 0, 0, 0, 0, -\frac{243}{608} 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (15x + 4), 0, 0, 0, 0, -\frac{243}{608} 45^x \Gamma\left(x + \frac{4}{15}\right) \Gamma\left(x + \frac{2}{3}\right) (15x + 4) \right]$$

**(3.4)**