On the integrability of an ODE system with an inhomogeneous right-hand side

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Integrability

For an autonomous ODE system

$$\frac{d\,x_i(t)}{d\,t} = \phi_i(x_1(t), \ldots, x_n(t)), \quad i = 1, \ldots, n,$$

we call the first integral the differentiable function $I_k(x_1(t), \ldots, x_n(t))$, whose derivative in the direction of the vector field vanishes

$$\left.\frac{d\,I_k(x_1(t), \ldots, x_n(t))}{d\,t}\right|_{\frac{d\,x_i(t)}{d\,t} = \phi_i(x_1(t), \ldots, x_n(t))} = 0, \quad k = 1, \ldots, m.$$

- We call the system integrable if the number of such independent first integrals m is large enough.
- For a two-dimensional system to be integrable, it is sufficient to have one first integral. We also add the requirement that this integral is a real function.

A simple example

- Consider the equation of the harmonic oscillator

$$\ddot{x}(t) + \omega_0^2 \cdot x(t) = 0.$$

- This equation is equivalent to the system

$$\begin{cases} \dot{x}(t) = & y(t), \\ \dot{y}(t) = & -\omega_0^2 \cdot x(t). \end{cases}$$

- The first integral of such a system will be

$$I(x(t), y(t)) = x^2(t) + y^2(t)/\omega_0^2.$$

- Indeed, its time derivative is zero due to the equations of the system above

$$\frac{d\, I(x(t), y(t))}{d\, t} = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)/\omega_0^2 = 2x(t)y(t) - 2x(t)y(t) = 0$$

Solution of the integrable system

From the constancy of the first integral along the trajectory

$$I(x(t), y(t)) = C_1^2,$$

we obtain

$$y(t) = \pm\omega_0\sqrt{C_1^2 - x^2(t)}.$$

By eliminating $y(t)$, we reduce the dimension of the system

$$\frac{dx(t)}{dt} = \pm\omega_0\sqrt{C_1^2 - x^2(t)}$$

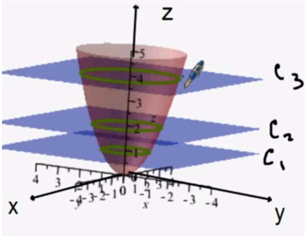
and solve this system in quadratures, becouse the variables of the system are divisible

$$\frac{dx(t)}{\sqrt{C_1 - x^2(t)}} = \pm\omega_0 \cdot dt, \quad \text{i.e. } \arcsin(x(t)/C_1) = \pm\omega_0 \cdot t + C_2.$$

Finally, we obtain $\quad x(t) = C_1 \cdot \sin(\pm\omega_0 \cdot t + C_2).$

- Integrability is a very useful property. In particular, an integrable system is solvable in quadratures, i.e. in the form of an analytical expression, which is important for constructing computer algebra algorithms.
- Knowledge of first integrals is very useful in constructing symplectic schemes of numerical integration, for studying phase portraits, etc.

  For example, the presence of a local extremum in a real integral indicates the presence of closed level lines, i.e. periodic solutions (Lyapunov):



Problem statement

The problem is to find such values of the parameters of a nonlinear dynamical system that make it integrable.

- This problem arose in connection with the study of the integrability of a two-dimensional autonomous system near a strongly degenerate singular point [Bruno, Edneral 2009], [Bruno, Edneral 2024].
- Similarly, the problem is posed in the papers

  “Integrability of complex planar systems with homogeneous nonlinearities” [Ferčec, Giné, Romanovski, Edneral 2016] and “On the Integrability of Persistent Quadratic Three-Dimensional Systems” [Ferčec , Žulj, Mencinger 2024].

  These works are based on the resolution of algebraic conditions for the existence of an integrating factor or Darboux integrals. The papers study resonance cases of systems with exclusively homogeneous nonlinearity in the right-hand sides, which determined the title of today’s report.
- Our approach is based on local analysis, considering local integrability.

Definition of local integrability

Consider an autonomous system of ordinary differential equations

$$\frac{d\,x_i}{d\,t} = \phi_i(X), \quad i = 1, \ldots, n,$$

where $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ are dependent variables and $\phi_i(X)$ are polynomials.

In a small neighborhood of a point $X = X^0$, the system *is locally integrable* if it has there a sufficient number m of independent first local (or formal) integrals of the form

$$I_k(X) = \frac{a_k(X)}{b_k(X)}, \quad k = 1, \ldots, m,$$

where $a_k(X)$ and $b_k(X)$ are analytic functions in a neighborhood of X^0. In the two-dimensional case, the existence of one formal integral is sufficient for integrability.

Regular and singular points

- A regular point of an autonomous system of ODEs of dimension n

$$\frac{dx_i}{dt} = P_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n,$$

  is a point in the phase space with coordinates $(x_1^0, \ldots, x_n^0)$, in the neighborhood of which the functions P_i on the right-hand side have no singularities and do not vanish simultaneously.
- If any of the functions P_i is singular at this point, or all the functions vanish simultaneously there, then this point is called singular.
- When the right hand side is simultaneously zeroed at some point, it is called a stationary (or unmoved, equilibrium) point, since all time derivatives are zero there.
- If the functions P_i are polynomials, then all singularities of such systems are reduced to stationary points.

Integrability in a neighborhood of a regular point

In a small neighborhood of a regular point with coordinates $\{x_1^0, \dots, x_n^0\}$ the ODE system will look like this

$$\frac{dx_i}{dt} = P_i(x_1^0, \dots, x_n^0), \qquad i = 1, \dots, n,$$

i.e. the right-hand side will consist of constants.

- In the case of a single nonzero value among the right-hand sides, say $P_1(x_1^0, \dots, x_n^0) \neq 0$, all variables $x_2, \dots, x_n$ will be local integrals. So the system has necessary for the integrability of an autonomous system the $n-1$-th first integral.
- If a pair or more numbers $P_1(x_1^0, \dots, x_n^0)$ and $P_2(x_1^0, \dots, x_n^0)$ are nonzero, the first integrals will be expressions of the type

$$I(t) = P_1(x_1^0, \dots, x_n^0) \cdot x_2(t) - P_2(x_1^0, \dots, x_n^0) \cdot x_1(t) + Const.$$

Local Integrability

- Thus, in the neighborhood of a regular point it is always possible to write down a sufficient number of first integrals, which, however, will be integrals only in a small neighborhood of this point. Such first integrals are called local or formal, and the property they provide is local integrability.
- The solution of the system in the neighborhood of a regular point is trivial.

Multi-index notation

To study local integrability in the neighborhood of a stationary point, we use the absence of constant terms in the right-hand sides there, so we can write

$$\dot{x}_i = \lambda_i x_i + \sum_{j>i}^{n} a_{i,j} x_j + x_i \sum_{\mathbf{q}\in\mathcal{N}_i} f_{i,\mathbf{q}} \mathbf{x}^{\mathbf{q}}, \quad i = 1, \dots, n,$$

where we assumed an upper triangular shape of the linear part and used multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} {x_j}^{q_j}$$

with vector power $\mathbf{q} = (q_1, \dots, q_n)$. The i-th component of the vector $\mathbf{q}$ can be negative because we have moved the factor x_i out of the sum, so

$$\mathcal{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \ge -1 \text{ and } q_j \ge 0\,,\ \text{if } j \ne i\,, \quad j = 1, \dots, n\}\,.$$

Original system of equations

Let us write the system in the neighborhood of a singular point and diagonalize its linear part by a linear transformation

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q} \in \mathcal{N}_i} f_{i,\mathbf{q}} \mathbf{x}^{\mathbf{q}}, \quad i = 1, \ldots, n,$$

Note that the linear part can be non-diagonalizable only if there are identical eigenvalues. Note also that the normalizing transformation is also written for the general case of a triangular matrix of linear form, see [Bruno 1971,1972,1979,1989].

Normalizing transformation

We define normalizing transformation as a formal invertible quasi-identity change of variables

$$x_i \;=\; z_i + z_i \sum_{\mathbf{q} \in \mathcal{N}_i} h_{i,\mathbf{q}} \mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n,$$

as a result of which the original system is reduced to the normal form.

Note that such a transformation does not change the linear part of the system.

Normal form of the system

The normal form of the system

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q}, (\lambda_1, \ldots, \lambda_n) \rangle = 0 \\ \mathbf{q} \in \mathcal{N}_i}} g_{i,\mathbf{q}} \mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n,$$

is (almost) linear when the blue condition above does not hold in the so-called non-resonance case and is thus integrable.

Remember that all elements of the vector **q** are integers, so if, for example, $\lambda_1 = \sqrt{2}, \lambda_2 = 1$, then the scalar product there will never be zero.

For some eigenvalues, however, the right-hand side may additionally contain a power term.

Computing the normal form

The coefficients h and g in the normalization formulas can be computed to a given order using the recurrence relation

$$g_{i,\mathbf{q}} + \langle \mathbf{q}, (\lambda_1, \ldots, \lambda_n) \rangle \cdot h_{i,\mathbf{q}} = -\sum_{j=1}^{n} \sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q} \\ \mathbf{p},\mathbf{r} \in \bigcup_i \mathcal{N}_i \\ \mathbf{q} \in \mathcal{N}_i}} (p_j + \delta_{ij}) \cdot h_{i,\mathbf{p}} \cdot g_{j,\mathbf{r}} + \tilde{\Phi}_{i,\mathbf{q}} \,,$$

The following programs were written for this purpose:

- in STANDARD LISP [Edneral, Khrustalev 1992];
- in the upper level of the MATHEMATICA system language [Edneral, Khanin 2002].

Conditions for convergence of the normalizing transformation

Two conditions are formulated [Bruno 1971,1972]:

- Condition **A**. The coefficients of the normal form satisfy the relation

$$\sum_{\substack{\langle \mathbf{q}, (\lambda_1, \ldots, \lambda_n)\rangle = 0 \\ \mathbf{q} \in \mathcal{N}_i}} g_{i,\mathbf{q}} \mathbf{z}^{\mathbf{q}} = \lambda_i a(Z) + \bar{\lambda}_i b(Z), \quad i = 1, \ldots, n,$$

  where $a(Z)$ and $b(Z)$ are some formal power series independent of the index i.

- Condition ω (on small denominators). This condition is satisfied for almost all eigenvalues of λ_i. In any case, it is satisfied for rational values.

Theorem (Bruno 1971,1972)

If the eigenvalue vector $(\lambda_1, \ldots, \lambda_n)$ satisfies condition ω and the normal form satisfies condition **A** *then the normalizing transformation converges.*

Normal form under condition **A**

Let's consider the case of *N* : *M* resonance, i.e. $\lambda_2/\lambda_1 = -N/M$ in a two-dimensional system. Then the condition **A** there will look like this

$$N \cdot g_1(Z) = -M \cdot g_2(Z).$$

In this case, the normalized system can be rewritten as

$$N \times \left| \frac{d\ \log(z_1)}{d\,t} = \lambda_1 + g_1(Z)\ , \quad M \times \right| \frac{d\,log(z_2)}{d\,t} = -\frac{N}{M}\lambda_2 - \frac{N}{M} g_2(Z)\ .$$

Hence $\quad \dfrac{d\ \log(z_1^N \cdot z_2^M)}{d\,t} = 0,\ $ or $\ z_1^N \cdot z_2^M = const.$

Thus, this is a local first integral (formal). That is, the system is locally integrable.

Increasing the dimension of the system increases the dimension of the condition **A** and, as a consequence, the number of local first integrals.

So, in a neighborhood of the stationary point, the condition **A**:

- Ensures convergence of the normalizing transformation;
- Ensures local integrability at this point;
- Isolates periodic orbits if the eigenvalues are purely imaginary.

How can global and local integrability be related?

Thus, we study the integrability in a certain region of the phase space of an autonomous ODE system with a polynomial right-hand side. We propose a hypothesis [Edneral 2023]:

Hypothesis

Local integrability in the neighborhood of each point of some region of the phase space is necessary for the existence of a first integral there.

Indeed, if there is a continuous global first integral, then in the neighborhood of each point it can be considered as local.

Based on the formulated hypothesis, an algorithmic method for studying integrability can be constructed.

The local integrability condition as a finite algebraic system

- By calculating the normal form at a fixed point (the origin) up to a certain order, we can obtain the condition **A** as a set of algebraic equations for the parameters of the system.
- Generally speaking, the condition **A** is an infinite sequence of equations. But by virtue of Hilbert's basis theorem [Hilbert 1890], it is equivalent to a finite system. Which one?
- Observing algebraic systems for cases of condition **A** for various ODE systems, we found that the obtained solutions of the system stop changing if the number of equations in the system is greater than a certain value. We observed a similar "saturation" for all considered ODE systems. On this basis, we assume that the first (lower) equations already form a basis.
- Proof of this phenomenon would allow us to move from machine estimates of cyclicity of polynomial systems to rigorous estimates [Edneral 1997].

The Liénard System

To demonstrate the method, we chose an autonomous ODE system with an inhomogeneous right-hand side, constructed based on the Liénard equation [Liénard 1928]

$$\frac{d^2 x}{d t^2} + P(x)\frac{d x}{d t} + Q(x) = 0.$$

In the original notation, this equation assumes that P is even and Q is odd. We will consider these functions to be polynomials without certain parity properties.
The case where $P(x)$ is a linear polynomial and $Q(x)$ is a cubic polynomial was studied in [Edneral 2023]. 7 two-parameter families for 5 free parameters were found, for which the system turns out to be integrable.

Liénard system with fourth-order right-hand side

Let's write the equation as a parameterized polynomial system

$$\frac{d\,x}{d\,t} = y,$$
$$\frac{d\,y}{d\,t} = (a_0 + a_1 x + a_2 x^2)\, y + b_1 x + b_2 x^2\, y + b_3 x^3 + b_4 x^4.$$

Here t is independent, $x(t)$ and $y(t)$ are dependent variables, and $a_0, \ldots, b_4$ are 7 free parameters. Our task is to determine for which values of the parameters the system has a first integral and, thus, is integrable.

This task was proposed by Prof. M.V. Demina [2020], who found the first integral for the case

$$\frac{d\,\tilde{x}(t)}{d\,t} = \tilde{y}(t),$$
$$\frac{d\,\tilde{y}(t)}{d\,t} = \left(\frac{25}{12} - 3\tilde{x}^2(t)\right)\tilde{y}(t) - \frac{125}{432} + \frac{25}{36}\tilde{x}(t) + 5\tilde{x}(t) + 7\tilde{x}^3(t) + 3\tilde{x}^4(t).$$

Local integrability condition

At all regular points the system is integrable. At all stationary non-resonant points it is also integrable. Restrictions on the parameters can arise only at a nearby point with resonance.

Resonance of the linear part $1 : M \neq 1$ occurs in the system under study if the relation

$$b_1 = \frac{a_0^2 M}{(M-1)^2} \quad \text{is satisfied.}$$

Fixing the parameter b_1 in this way, we then calculate the normal form up to $3(M+1)$-th order for various resonances, and obtain the first three algebraic equations of the local integrability condition in 6 variables. Attempts to solve these systems for various resonances using the MATHEMATICA system were unsuccessful (See the pdf file).

A special case of integrability

By shifting one of the fixed points to the origin, the above Demina's case can be rewritten as

$$\frac{d\,x(t)}{d\,t} = y(t),$$

$$\frac{d\,y(t)}{d\,t} = \left(2 - x(t) - 3x^2(t)\right) y(t) + 3x(t)(1 + x(t))^3.$$

Substituting the corresponding values of the parameters

$$a_0 = 2, a_1 = -1, a_2 = -3, b_1 = 3, b_2 = 9, b_3 = 9, b_4 = 3,$$

into the algebraic system corresponding to the $1:3$ resonance, satisfies it identically. Thus, we have at least one zero-dimensional solution and it indeed corresponds to the integrable case.

Generalization of Demina's case

It is interesting to search for families of solutions that include the found ones. To do this, we parameterize the system in a similar form

$$\frac{d\,x(t)}{d\,t} = y(t),$$

$$\frac{d\,y(t)}{d\,t} = (c_0 + c_1 x(t) + c_2 x^2(t))\,y(t) + d_1 x(t)(1 + d_2 x(t))^3.$$

The new parameters $c_0, \ldots, d_2$ are related to the current ones as follows

$$a_0 = c_0,\ a_1 = c_1,\ a_2 = c_2,\ b_1 = d_1,\ b_2 = 3d_1 d_2,\ b_3 = 3d_1 d_2^2,\ b_4 = d_1 d_2^3,\ \text{here}\quad d_1 = \frac{3}{4}c_0^2. \tag{1}$$

Replacing the variables in the algebraic system for the $1:3$ resonance using the formulas (1), we obtain an algebraic system of three equations for the parameters c_0, c_1, c_2, d_2.

A new case of integrability

Using the Solve procedure of the MATHEMATICA-11 system, we can obtain its rational solutions

$$\{c_1 = 0, c_2 = 0, d_2 = 0\},\ \{c_1 = -\frac{1}{2} c_0\, d_2,\ c_2 = -\frac{3}{2} c_0\, d_2^2\}.$$

The first of these solutions reduces the system to a linear integrable form and is not interesting, the second gives a system with two free parameters c and d (we removed the indices from the parameters)

$$\frac{d\,x(t)}{d\,t} = y(t),$$
$$\frac{d\,y(t)}{d\,t} = \frac{c}{4}(d\;x(t)+1)[3c\;x(t)(d\;x(t)+1)^3 + 2y(t)(3d\;x(t)-2)].$$

This system is integrable since it has a first integral of the form

$$I(x(t), y(t)) = \sqrt[3]{c\,x(t)(d\,x(t)+1)^2 + 2y(t)} \left(\frac{c\,(d\,x(t)+1)^2\ {}_2F_1\left(1, 1; \frac{5}{3}; -\frac{c\,(d\,x(t)+1)^3}{2d\,y(t) - c(d\,x(t)+1)^2}\right)}{c\,(d\,x(t)+1)^2 - 2d\,y(t)} + 2 \right).$$

Here ${}_2F_1$ – hypergeometric function [Bateman, Erdelyi 1953].

Conclusions

- For a two-dimensional autonomous polynomial ODE system, an algorithm for constructing an algebraic system on ODE parameters such that first integrals can be found on families of its solutions is proposed and implemented.
- The algorithm is based on the study of resonances at stationary points of the ODE system.
- A hypothesis is proposed that for the integrability of an autonomous polynomial ODE system in a certain region of the phase space, local integrability at each point of this region is necessary. Integrability means the existence of a differentiable function of the system variables whose derivative in the direction of the vector field is zero.

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Thank you for your attention