Relative primality and other properties of trinomials

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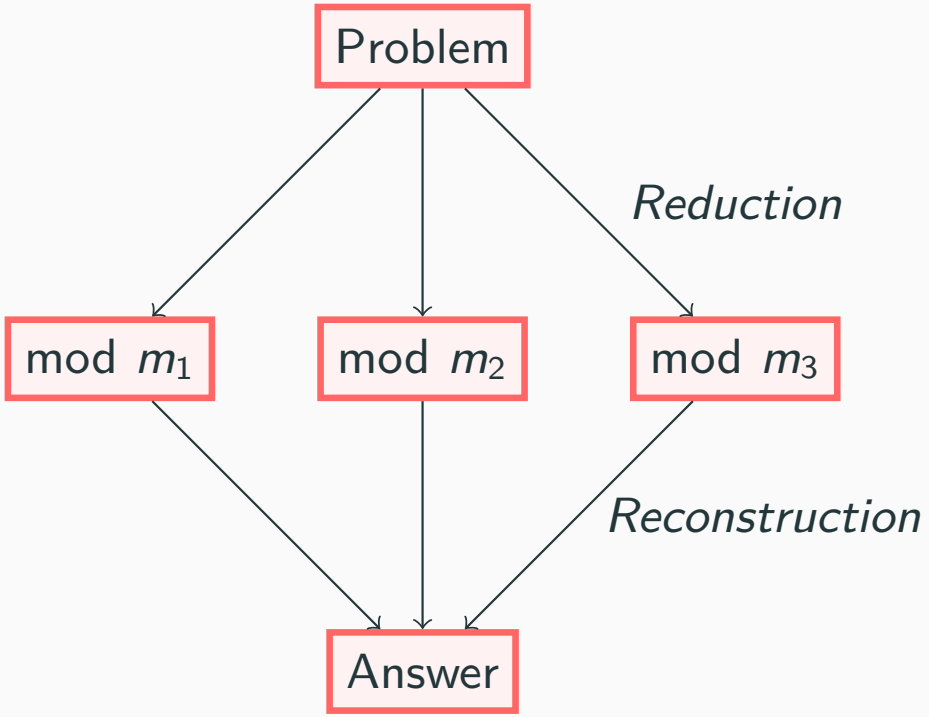
Joint work with Mits Kobayashi, Natalya Ter–Saakov, and Eugene Zima

Computations are often sped up with the Chinese Remainder Theorem.

Criteria for "good" moduli:

- Easy to find
- Fast to reduce and reconstruct
- Roughly the same size (balanced)



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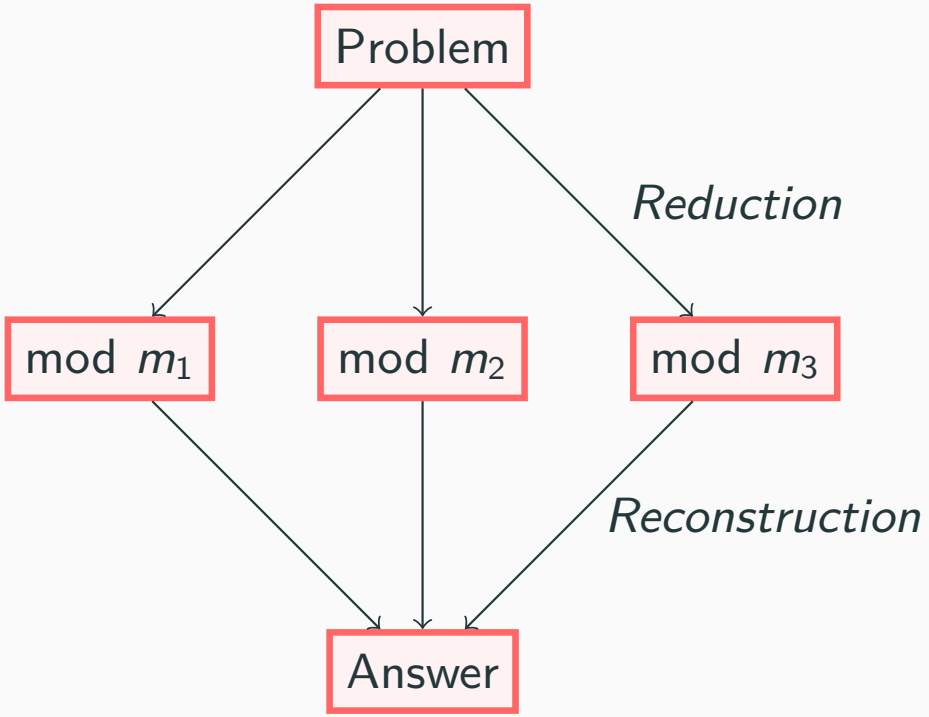

Chen and Zima (2023) proposed "trinomial" moduli:

$$2^n - 2^k + 1 \quad (n \text{ is fixed}, 0 < k < n)$$

These are perfectly balanced and have fast reduction / reconstruction.

Fast reconstruction

Imagine Chinese Remaindering with these moduli:

$$m_1 = 2^5 - 2^1 + 1$$
$$m_2 = 2^5 - 2^3 + 1.$$

Reconstruction of x looks like this:

$$A = x \bmod m_1$$
$$B = x \bmod m_2$$

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$$B = x \bmod m_2 \qquad \Longrightarrow$$

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Reconstruction of x looks like this:

$$\begin{array}{lll} A = x \bmod m_1 & x = x_0 + x_1 m_1 & x_0 = A \\ B = x \bmod m_2 & \Longrightarrow & x_1 = (B - x_0) m_1^{-1} \pmod{m_2} \end{array}$$

By chance,

$$m_1^{-1} \bmod m_2 = -2^2,$$

so the multiplication by $m_2^{-1} \bmod m_2$ is very fast.

There is more to these moduli.

$$m_1 = 2^5 - 2^1 + 1$$
$$m_2 = 2^5 - 2^3 + 1$$
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scale exponents by c

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$$m_1 = 2^{5c} - 2^c + 1$$
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We can make c very large, but the reconstruction step

$$\begin{aligned} x_1 &= (B - x_0) m_1^{-1} \pmod{m_2} \\ &= -2^{2c}(B - x_0) \pmod{m_2} \end{aligned}$$

is always a bitshift.

Reconstruction is linear time (relative to the bitlength of moduli).

Nice inverses do not always happen.

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Good moduli

$$2^5 - 2^1 + 1$$

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Inverses come from polynomial identity:

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Bad moduli

$$2^6 - 2^2 + 1$$
$$2^6 - 2^5 + 1$$

GCD equation:

$$(x^6 - x^2 + 1)(-x^5 + 4x^4 - 5x^3 + x^2 - 3x + 2) + (x^6 - x^5 + 1)(x^5 - 3x^4 + 2x^3 + x^2 + 3x + 5) = 7.$$

The 7 ruins us!

Definition

$x^n - x^k + 1$ and $x^n - x^j + 1$ *dyadically resolve* if their resultant is a signed power of 2.

The inverse sequence

$$(2^{cn} - 2^{ck} + 1)^{-1} \bmod (2^{cn} - 2^{cj} + 1)$$

will be "nice" if and only if $x^n - x^k + 1$ and $x^n - x^j + 1$ dyadically resolve.

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Basic questions (new)

1. When do $x^n - x^k + 1$ and $x^n - x^j + 1$ dyadically resolve?
2. Are there arbitrarily large sets of dyadically resolving trinomials?
3. How can we efficiently find these sets?

Definition

Let $T(n)$ be the graph with vertices $\{1, 2, 3, \ldots, n-1\}$ that contains the edge $\{k, j\}$ if and only if $x^n - x^k + 1$ and $x^n - x^j + 1$ dyadically resolve.

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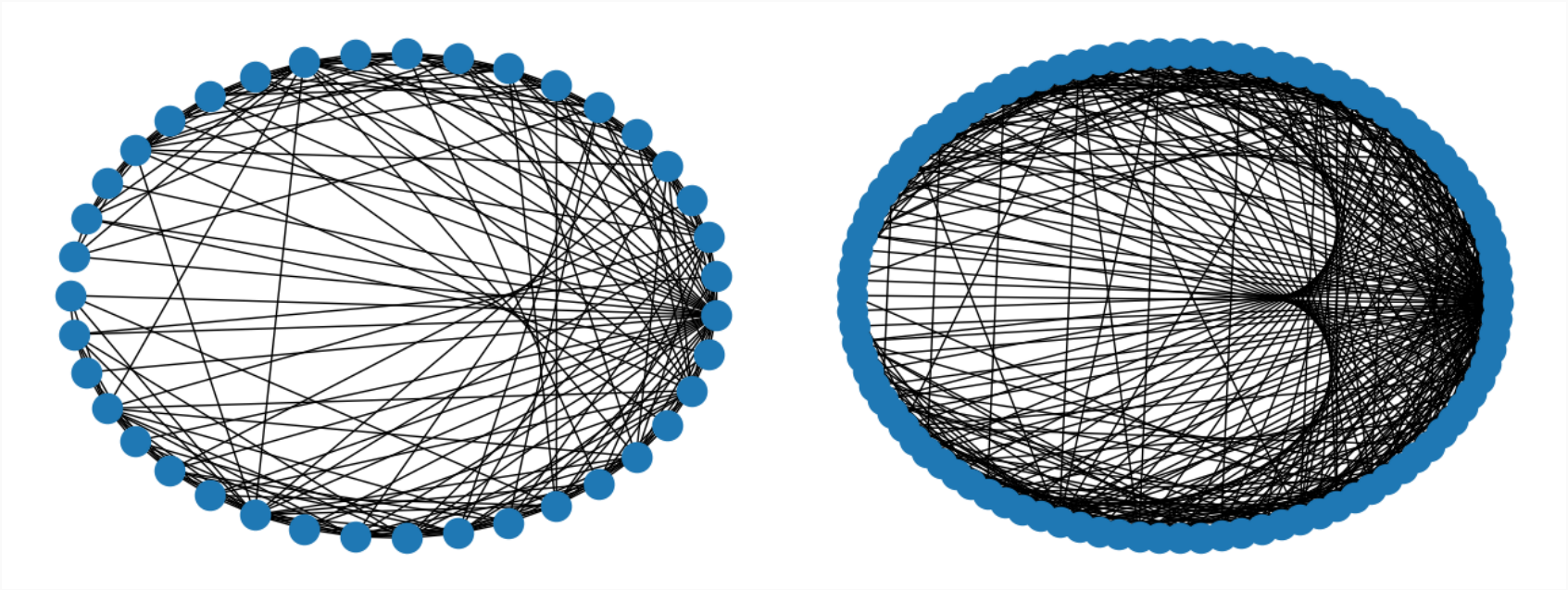
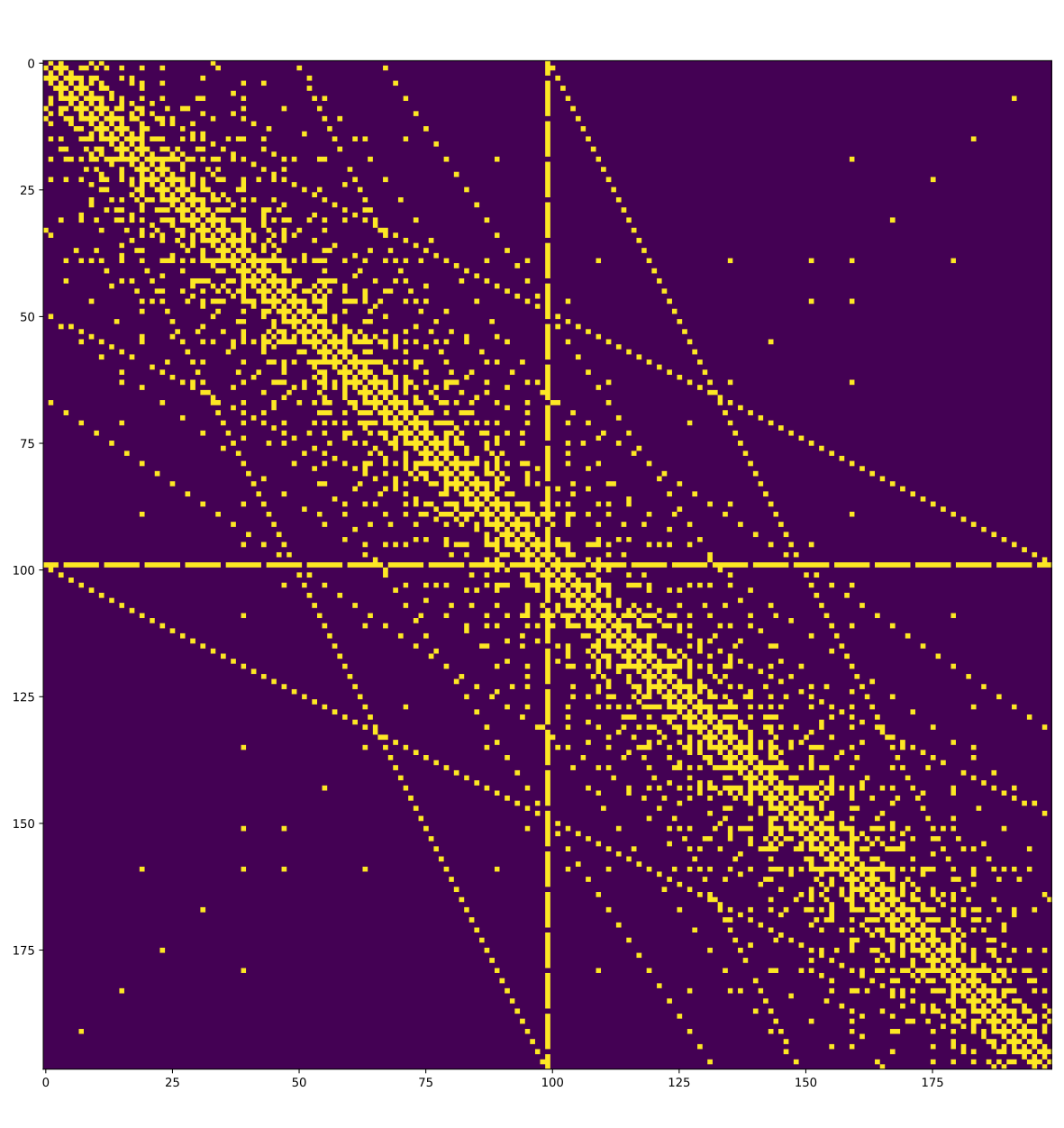


$T(40)$ and $T(100)$

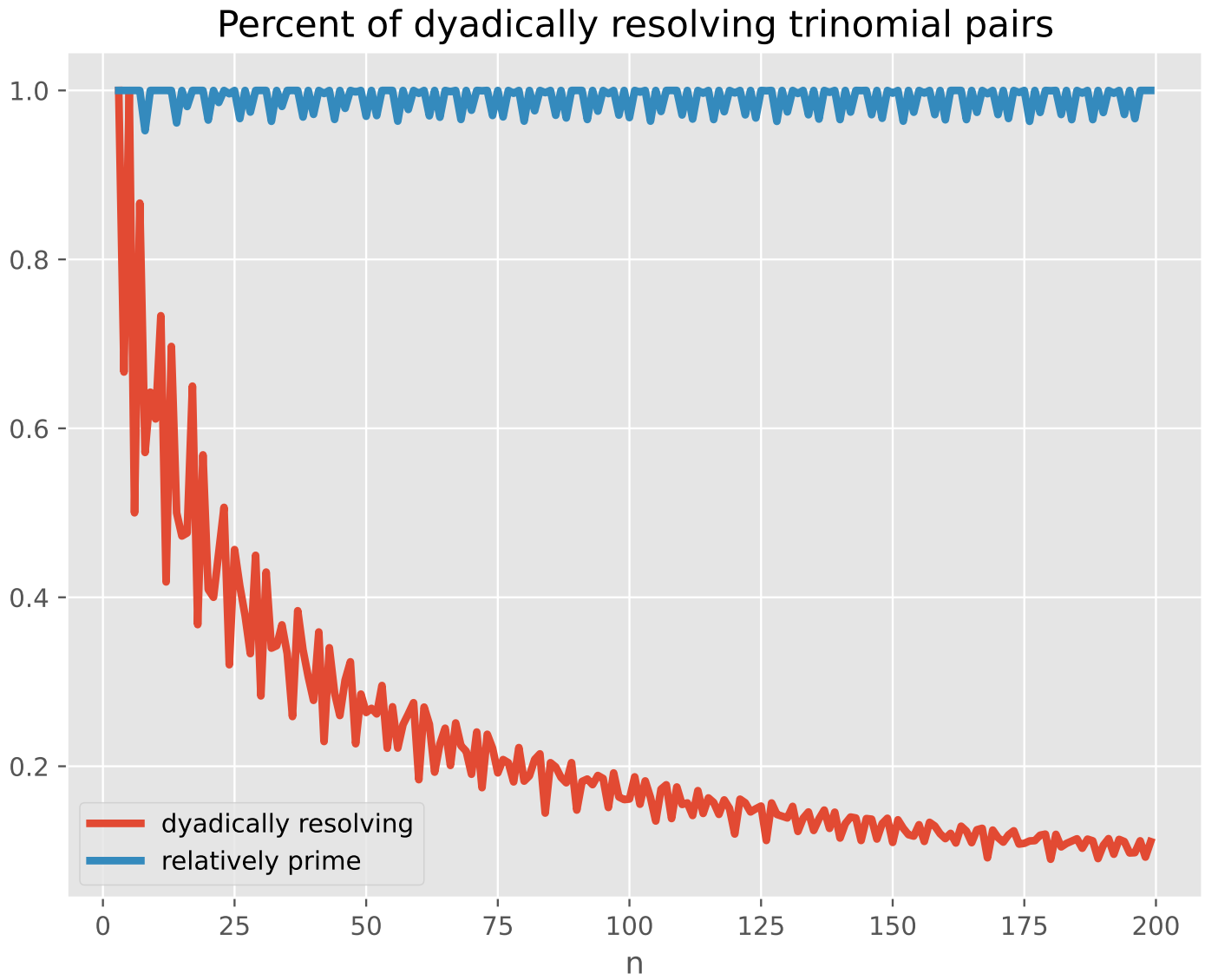
Question 1

When is $\mathrm{res}(x^n - x^k + 1, x^n - x^j + 1)$ a signed power of 2?

What are the edges of $T(n)$?



Adjacency matrix of $T(200)$.



Very few powers of 2! But lots of relatively prime pairs?

Theorem (RDB, Kobayashi, Ter-Saakov, Zima)

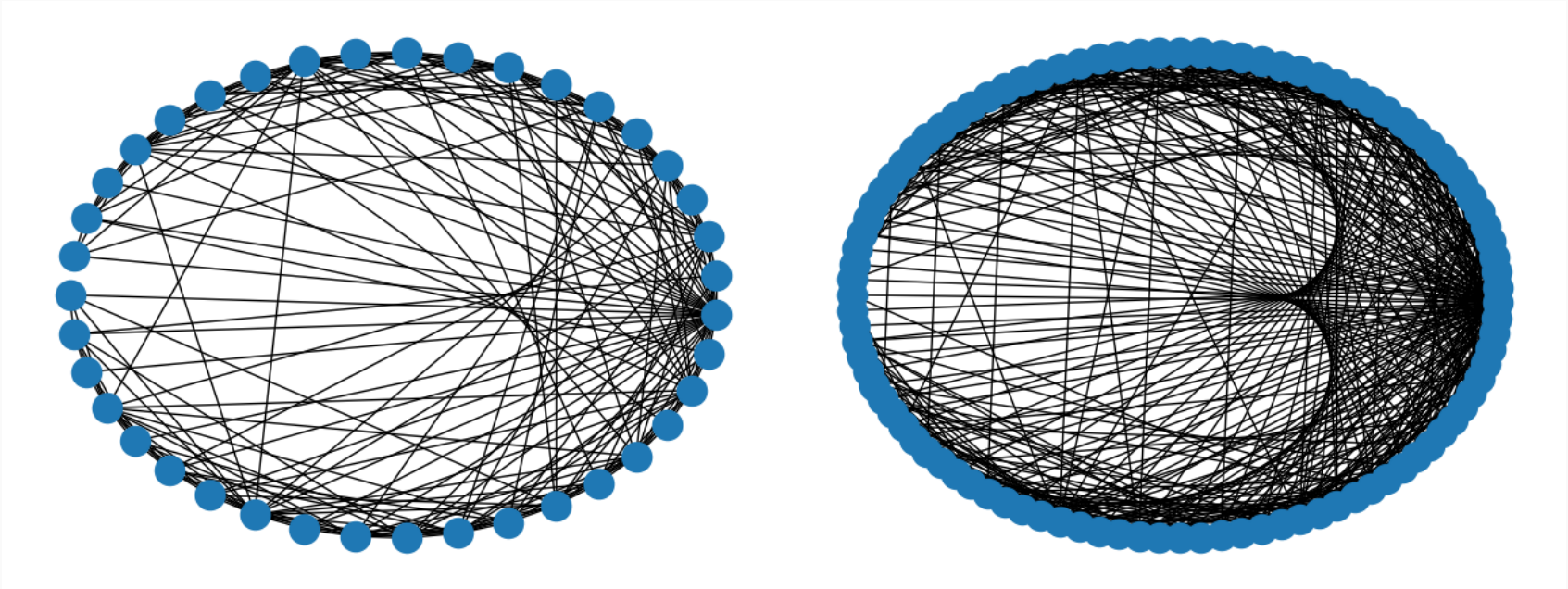
If $g(x) := \gcd(x^n - x^k + 1, x^n - x^j + 1) \neq 1$, *then:*

- n *is even;*
- $k - j$ *is divisible by* 6*; and*
- $g(x)$ *is a product of cyclotomic polynomials whose orders are multiples of* 6*.*

Approximately 97% of all pairs of trinomials for large n are relatively prime.

We have no corresponding statement for dyadically resolving pairs.

The proportion probably goes to 0.



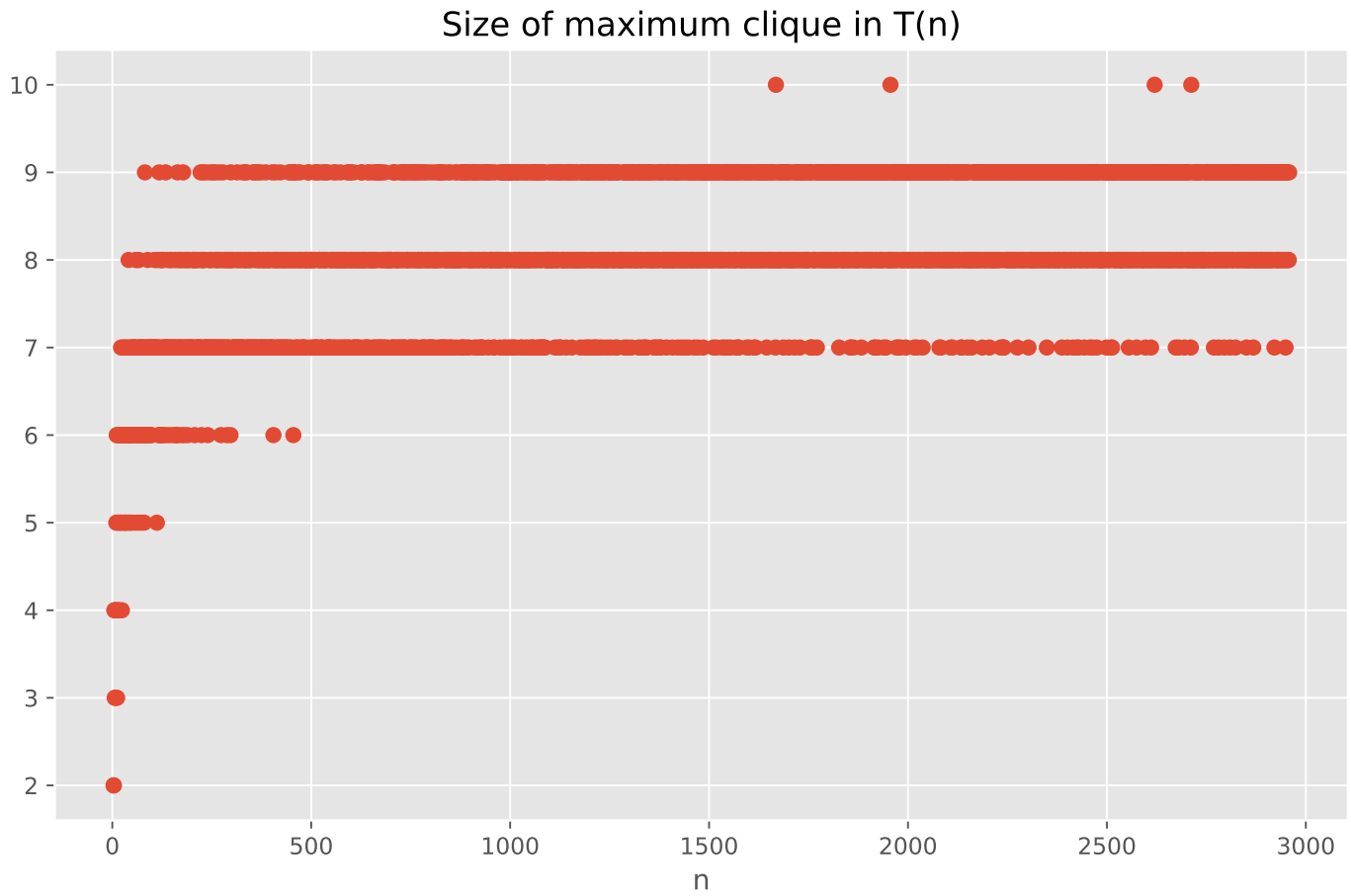
$T(40)$ and $T(100)$

Questions 2 and 3

What is the largest set of *pairwise dyadically resolving* trinomials of degree n?

What is the largest clique in $T(n)$?

Computing maximum cliques is fast!



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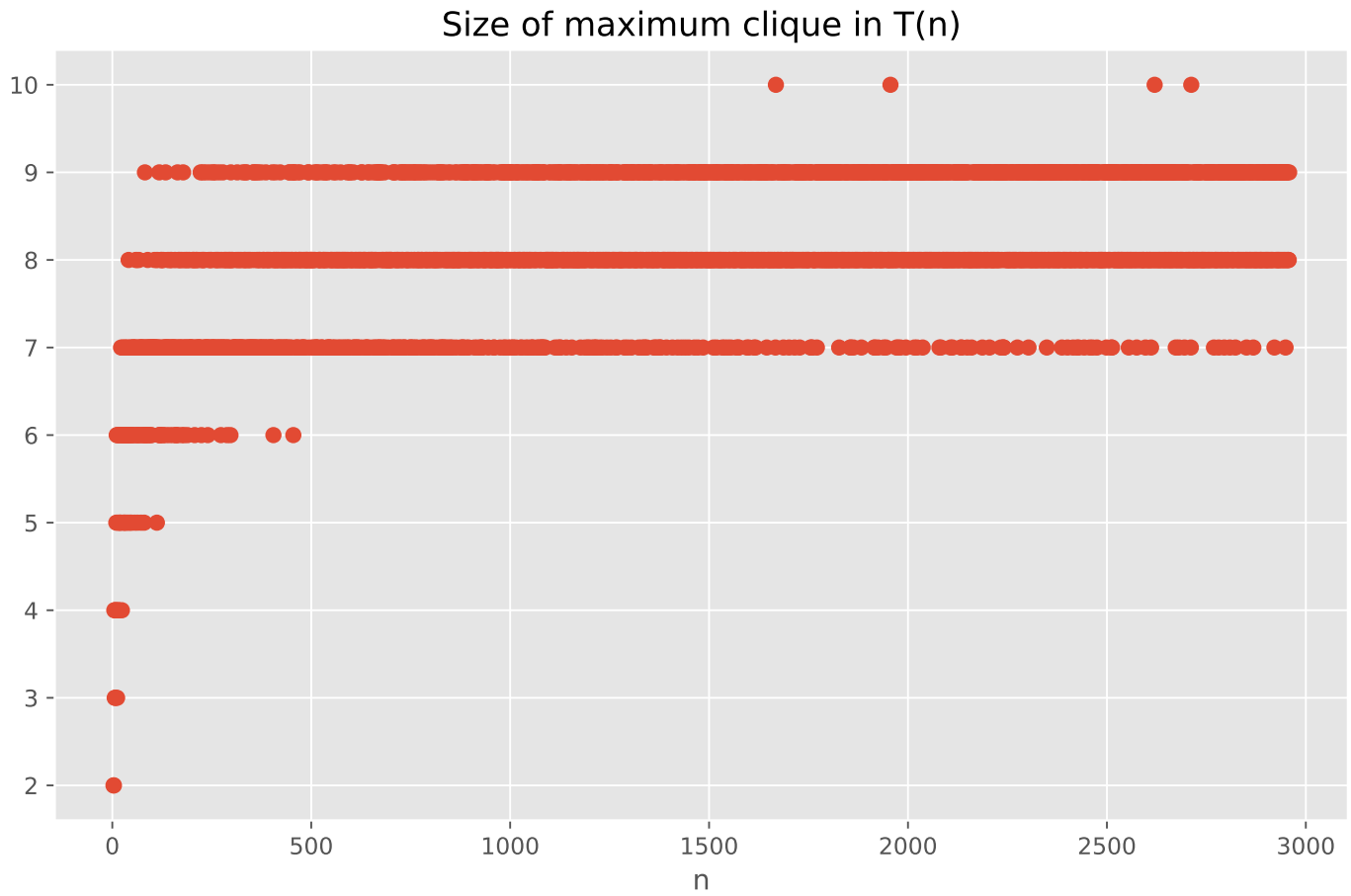

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Theorem

The size of the largest clique in $T(n)$ goes to ∞ as $n \to \infty$.

We do not know the true growth rate of the largest cliques.

We have not found a *reasonable* clique of size 11.

clique size k	smallest n
2	3
3	5
4	5
5	10
6	11
7	22
8	41
9	82
10	1668
11	> 3000

The best estimate we have is the following.

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$v_2(n) = v$ is the largest v such that 2^v divides n.

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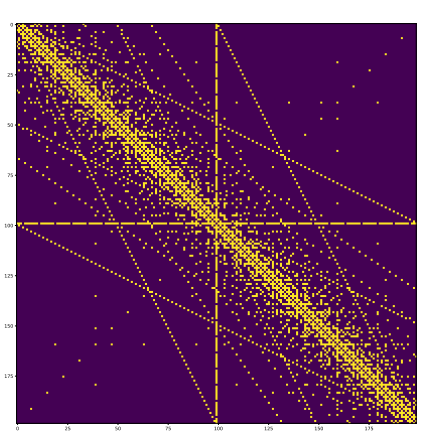
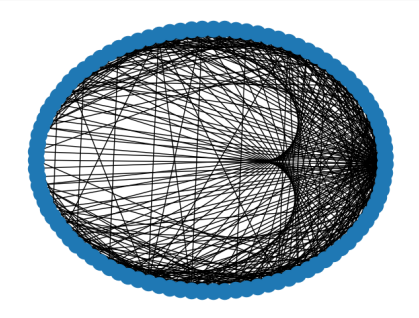
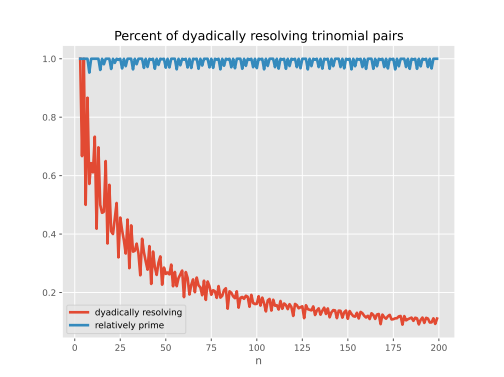
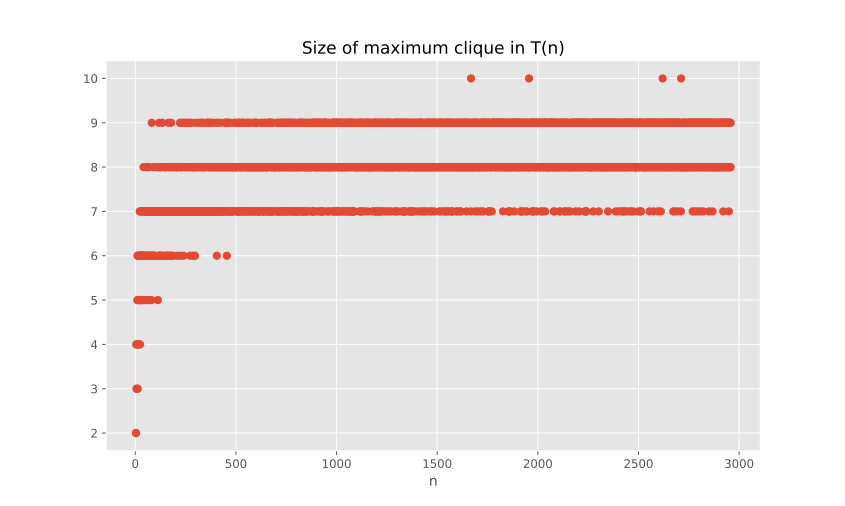
Open questions









True growth of largest clique sizes?

Edge density of $T(n)$?

Other moduli shapes: $2^n \pm 2^k \pm 1$, ...