On Truncated Series Involved in Exponential-Logarithmic Solutions of Truncated LODEs

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Suppose that K is an algebraically closed field of characteristic 0.
For a linear ordinary differential equation

$$a_r(x)y^{(r)}(x) + a_{r-1}(x)y^{(r-1)}(x) + \cdots + a_0(x)y(x) = 0 \quad (1)$$

with $a_i(x) = \sum_{j=0}^{\infty} a_{ij}x^j \in K[[x]], i = 0, 1, \ldots, r$, its formal *exponential-logarithmic* solutions are solutions in the form

$$e^{Q(x^{-1/q})}\, x^{\lambda} w(x^{1/q}), \quad (2)$$

where Q is a polynomial with coefficients in K, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$,

$$w(x) = \sum_{s=0}^{m} w_s(x) \ln^s x,$$

$m \in \mathbb{Z}_{\geq 0}$, $w_s(x) \in K((x))$, $s = 0, \ldots, m$, and $w_m(x) \neq 0$.

In (2), the factor $x^{\lambda} w(x^{1/q})$ is *the regular part*, $Q(x^{-1/q})$ is *the exponent of the irregular part*, and λ is *the exponent of the regular part*.

In our previous publications we have considered linear ordinary differential equations with coefficients given as truncated power series:

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j + O(x^{t_i+1}), \quad t_i \in \mathbb{Z}_{\geqslant 0}, \quad i = 0, 1, \ldots, r.$$

Each computed solution (2) involves a finite set of truncated series

$$w_s(x) = \sum_{j=j_s}^{k_s} w_{sj}x^j + O(x^{k_s+1})$$

$j_s, k_s \in \mathbb{Z}$, $w_{s,j_s} \neq 0$, $k_s \geqslant j_s$, $s = 0, 1, \ldots, m$, for which the maximum possible number k_s of terms is calculated.

Abramov S.A, Khmelnov D.E., Ryabenko, A.A. Truncated series and formal exponential-logarithmic solutions of linear ordinary differential equations. Computational Mathematics and Mathematical Physics, 2020. Vol. 60, pp. 1609–1620.

Factually, we get all the available information about these solutions, having in mind the information that is invariant with respect to possible prolongations of the truncated series which are the coefficients of the given equation.

Definition

A prolongation *of a truncated series is a series, possibly also truncated, whose initial terms coincide with the known initial terms of the original truncated series; correspondingly, a prolongation of an equation with truncated-series coefficients is an equation, whose coefficients are prolongations of the coefficients of the original equation.*

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The following is an example of a truncated equation:

$$\left(x^3 + \frac{x^5}{3} + \mathrm{O}(x^6)\right)y'(x) + \left(1 + 3x + \mathrm{O}(x^3)\right)y(x) = 0. \tag{3}$$

Two different prolongations of equation (3) are, for example:

$$\left(x^3 + \frac{x^5}{3} + 4x^6 + \mathrm{O}(x^7)\right)y'(x) + \left(1 + 3x + x^3 + \mathrm{O}(x^4)\right)y(x) = 0, \tag{4}$$

$$\left(x^3 + \frac{x^5}{3} - 4x^6 + \mathrm{O}(x^7)\right)y'(x) + \left(1 + 3x + \mathrm{O}(x^4)\right)y(x) = 0. \tag{5}$$

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On slide 5, equation (3) has the truncated solution

$$e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}\big(C+\mathrm{O}(x)\big), \tag{6}$$

where C is an arbitrary constant.

Equation (4), which is prolongation of (3), has the truncated solution

$$e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}\big(C+4Cx+\mathrm{O}(x^2)\big). \tag{7}$$

Equation (5), which is prolongation of (3) too, has the truncated solution

$$e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}\big(C-3Cx+\mathrm{O}(x^2)\big). \tag{8}$$

Solutions (7) and (8) are prolongations of (6) and they are different in the second terms of series.

So, solution (6) of equation (3) is maximum invariant.

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Algorithms for constructing maximum invariant truncated solutions and the implementation of the algorithms in Maple were presented in our previous works.

In other words, the presented algorithms provide *the exhaustive use of information on a given equation.*

Now we are focusing on the question of automatic confirmation of such exhaustive use of information on a given equation, i.e. the confirmation that it is not possible to add any additional terms to the constructed truncated solution that are invariant with respect to prolongations of the given equation.

In order to confirm this, we demonstrate *a counterexample* with different prolongations of the given equation which *lead to the appearance of different additional terms in the solutions.*

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Theorem 1

Let $\mathcal{E}$ be an equation with truncated series coefficients. Then there exist and can be constructed algorithmically two different prolongations $\bar{\mathcal{E}}$, $\bar{\bar{\mathcal{E}}}$ of $\mathcal{E}$ having the following property:

If s is such a truncated solution of $\mathcal{E}$ that any series involved in s has the maximum possible truncation degree then **all** *the truncated series involved in the corresponding solutions $\bar{s}$, $\bar{\bar{s}}$ of resp. $\bar{\mathcal{E}}$, $\bar{\bar{\mathcal{E}}}$ have pair-wise different additional terms.*

The automatic confirmation of exhaustive use of the information on a given equation in the truncated solution is implemented as an extension of `FormalSolution` procedure from our `TruncatedSeries` package in Maple.

Example

Consider the following equation with the truncated-series coefficients and construct its solution using the `TruncatedSeries` package. The differential equation is written using the operator $\theta = x\frac{d}{dx}$:

$$> eq := (x^4 + \mathrm{O}(x^7))\theta(y(x), x, 3) + (3x + \mathrm{O}(x^5))\theta(y(x), x, 2) + (1 + x + 2x^2 + 3x^3 + \mathrm{O}(x^4))\theta(y(x), x, 1) + \mathrm{O}(x^5)y(x) = 0 :$$

$$> TruncatedSeries:-FormalSolution(eq, y(x));$$

$$\left[\_c_1 + \mathrm{O}\left(x^5\right) + \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(\_c_2 + \frac{35\_c_2 x}{27} + \frac{8947\_c_2 x^2}{1458} + \mathrm{O}\left(x^3\right)\right) + \mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} y_{reg}(x)\right] \tag{9}$$

where $y_{reg}(x)$ denotes the regular part, which can have different exponent λ for different prolongations of the original equation.

As a result of running the *FormalSolution* command with the new optional argument '*counterexample*' = '*Eqs*', the variable *Eqs* will be assigned a pair of the equations which forms one of the possible counterexamples:

> $FormalSolution(eq, y(x), \texttt{'}counterexample\texttt{'} = \texttt{'}Eqs\texttt{'})$:

The first counterexample equation:

> *Eqs*[1];

$$\left(x^5 + \mathrm{O}\left(x^6\right)\right) y(x) + \left(1 + x + 2x^2 + 3x^3 + 4x^4 + \mathrm{O}\left(x^5\right)\right) \theta(y(x), x, 1)$$
$$+ \left(3x + \mathrm{O}\left(x^6\right)\right) \theta(y(x), x, 2) + \left(x^4 - 4x^7 + \mathrm{O}\left(x^8\right)\right) \theta(y(x), x, 3) = 0 \tag{10}$$

The second counterexample equation:

> *Eqs*[2];

$$\left(5x^5 + \mathrm{O}\left(x^6\right)\right) y(x) + \left(1 + x + 2x^2 + 3x^3 - 2x^4 + \mathrm{O}\left(x^5\right)\right) \theta(y(x), x, 1)$$
$$+ \left(3x + \mathrm{O}\left(x^6\right)\right) \theta(y(x), x, 2) + \left(x^4 - x^7 + \mathrm{O}\left(x^8\right)\right) \theta(y(x), x, 3) = 0 \tag{11}$$

Using *FormalSolution* we obtain the truncated solution:

> $FormalSolution(Eqs[1], y(x));$

$$\left[\_c_1 - \frac{\_c_1 x^5}{5} + \mathrm{O}\left(x^6\right)\right.$$

$$+ \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(\_c_2 + \frac{35\_c_2 x}{27} + \frac{8947\_c_2 x^2}{1458} + \frac{5779943\_c_2 x^3}{118098} + \mathrm{O}\left(x^4\right)\right)$$

$$\left. + \frac{\mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} \left(\_c_3 + \mathrm{O}(x)\right)}{x^{\frac{17}{3}}}\right] \tag{12}$$

> *FormalSolution*$(Eqs[2], y(x))$;

$$\Big[\_c_1 - \_c_1 x^5 + \mathrm{O}(x^6)$$

$$+ \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(\_c_2 + \frac{35\_c_2 x}{27} + \frac{8947\_c_2 x^2}{1458} + \frac{5858675\_c_2 x^3}{118098} + \mathrm{O}(x^4)\right)$$

$$+ \mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} x^{\frac{10}{3}} (\_c_3 + \mathrm{O}(x))\Big] \tag{13}$$

It is easy to see the difference between the truncated series in solutions (12), (13).

Outcome

So, in the previous publications, we presented an algorithm for finding exponential-logarithmic solutions of linear ordinary differential equations with coefficients in the form of series, for which only a finite number of initial terms is known. Each solution involves a finite set of power series, for which the maximum possible number of terms is calculated.

In the current talk, the algorithm is supplemented with the option to provide a visual confirmation the impossibility of obtaining a larger number of terms in the series without using additional information about the given equation. Such a confirmation has the form of a counterexample to the assumption that it is possible to obtain additional terms of the series involved in the solution that are invariant to all prolongations of the given equation.