Truncated and Infinite Power Series in the Role of Coefficients of Linear Ordinary Differential Equations

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Infinite power series play an important role in mathematical studies. Those series may appear as inputs for many of mathematical problems.

In order to be able to discuss the corresponding algorithms and programs, we must agree on representation of the infinite series (algorithm inputs are always objects, represented by specific finite words in some alphabet).

We use below two possibilities for representing power series.

We write θ for $x\frac{d}{dx}$ and consider differential equations of the form

$$a_r(x)\theta^r y + a_{r-1}(x)\theta^{r-1} y + \cdots + a_0(x)y = 0 \qquad (1)$$

where y is an unknown function of x.

The ground field K is supposed to be a constructive field of characteristic 0.

For each $a_i(x)$, $i = 0, 1, \ldots, r$, one of two possibilities is allowed: $a_i(x)$ can be

- **an algorithmically represented series**: $a_i(x) = \sum_{j=0}^{\infty} \Xi_i(j)x^j$, where an algorithm $\Xi_i(j)$ is a procedure, terminating in finitely many steps;

  or
- **a truncated series**: $a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j + O(x^{t_i+1})$, where t_i is *the truncation degree* of the series.

In the frame of one-unique equation both types of coefficients may appear.

Definition 1

An expression of the form

$$f(x) + O(x^{k+1}), \tag{2}$$

in which $f(x) \in K[x^{-1}, x] \setminus \{0\}$ *and* k *is an integer* $\geq \deg f(x)$, *is called a* truncated Laurent solution *of (1), if for any specification of all* $O(x^{t_i+1})$ *(that is, for any replacement of the symbols* $O(\ldots)$ *by concrete series having corresponding valuation) included in the coefficients* $a_i(x)$ *of equation (1), such a specification of the series* $O(x^{k+1})$ *in (2) is possible that the specified expression (2) becomes a Laurent solution to the specified equation. (In other words, the solution (2) is* invariant *with respect to any prolongation of coefficients of the initial equation.) This* k *in (2) is* the truncation degree of the solution.

EQUATION THRESHOLD CONCEPT

Definition 2

Consider the set N of all integers n such that the equation (1) has a truncated Laurent solution whose truncation degree is n. Let N be nonempty and have the maximal element. We will call this element the threshold *of the equation (1). If the set N contains arbitrarily large integers, then we say that the threshold of the equation is ∞. If this set is empty, then the threshold is conventionally $-\infty$.*

Proposition 1

There exists no algorithm that, for an arbitrary equation (1), finds out whether its threshold is finite or infinite.

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Our current study is a continuation of the studies started in

- Abramov, S., Barkatou, M., Pflügel, E.: Higher-order linear differential systems with truncated coefficients. In: CASC 2011, LNCS, vol. 6885, 10–24 (2011)
- Abramov, S., Barkatou, M., Khmelnov, D.: On full rank differential systems with power series coefficients. Journal of Symbolic Computation **68**, 120–137 (2015)
- Abramov, S., Khmelnov, D., Ryabenko, A.: Procedures for searching Laurent and regular solutions of linear differential equations with the coefficients in the form of truncated power series. Progr. and Comp. Soft. **46**(2), 67–75 (2020)

in which it was assumed that either all the coefficients of a differential equation are represented algorithmically or all the coefficients are represented in the truncated form. In the current talk, the presence of both types of coefficients is allowed.

The presence of infinite series in the input data of a problem is a source of difficulties (the algorithmic impossibility of answering certain natural questions) — see Proposition 1. However, concerning our problem, we are faced, with a lighter version of the algorithmic undecidability. This undecidability is, so to speak, not too heavy.

Although we cannot indicate the greatest degree of the truncated Laurent solution existing for a given equation (the threshold of the equation). However, if we are interested in all solutions of a truncation degree not exceeding a given integer k then there is an algorithm, which allows us to construct all of them.

We developed and implemented such algorithm in Maple 2020 as `LaurentSolution` procedure from the package `TruncatedSeries`. The implementation and a session of Maple with examples of using `LaurentSolution` are available at

http://www.ccas.ru/ca/truncatedseries

EXAMPLES

```
> eq1:=(-1+x+x^2+O(x^3))*theta(y(x), x, 2)+
>       (-2+O(x^3))*theta(y(x), x, 1)+
>       (1+x+Sum(x^i/i!, i = 2 .. infinity))*y(x);
```

$$eq1 := \left(-1 + x + x^2 + O\left(x^3\right)\right)\theta(y(x), x, 2)$$

$$+\left(-2 + O\left(x^3\right)\right)\theta(y(x), x, 1) + \left(1 + x + \sum_{i=2}^{\infty} \frac{x^i}{i!}\right) y(x)$$

```
> LaurentSolution(eq1, y(x), 'threshold' = 'h');
```

$$[\,]$$

The equation has no truncated Laurent solutions.The threshold:

```
> h;
```

$$-\infty$$

It confirms that there are no such solutions.

```
> f:=proc(i)
>     if i = 0 then 0;
>     elif i::posint then
>       limit(diff(exp(-1/x^2), x $ i), x = 0);
>     else
>       'procname'(i);
>     end if;
> end proc:
> eq2:=(-1+x+x^2+O(x^3))*theta(y(x), x, 2)+
>      (-2+O(x^3))*theta(y(x), x, 1)+
>      Sum(f(i)*x^i, i = 0 .. infinity)*y(x);
```

$$eq2 := \left(-1 + x + x^2 + O\left(x^3\right)\right)\theta(y(x), x, 2)$$
$$+\left(-2 + O\left(x^3\right)\right)\theta(y(x), x, 1) + \left(\sum_{i=0}^{\infty} f(i)x^i\right)y(x)$$

Remark that $f(i) = \left(\exp(-\frac{1}{x^2})\right)^{(i)}\big|_{x=0}$.

```
> LaurentSolution(eq2, y(x), 'top' = 2, 'threshold' = 'h2');
```

$$\left[\frac{\_c_1}{x^2} - \frac{4\_c_1}{x} + \_c_2 + O(x),\ \_c_2 + O\left(x^3\right)\right]$$

The solutions with valuations -2 and 0 and with different truncation degrees are found. The threshold:

```
> h2;
```

$$\mathrm{FAIL}$$

$k = 2$ does not exceed the threshold.

```
> eq3:=(-1+x+x^2+O(x^3))*theta(y(x), x, 2)+
>       (-2+O(x^3))*theta(y(x), x, 1)+(x+6*x^2)*y(x);
```

$$eq3 := \left(-1 + x + x^2 + O\left(x^3\right)\right)\theta(y(x), x, 2) + \left(-2 + O\left(x^3\right)\right)\theta(y(x), x, 1) + \left(x + 6x^2\right)y(x)$$

```
> LaurentSolution(eq3, y(x), 'top' = 2, 'threshold' = 'h3');
```

$$\left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x), \_c_2 + \frac{1}{3}x\_c_2 + \frac{5}{6}x^2\_c_2 + O\left(x^3\right)\right]$$

The solutions with valuations -2 and 0 and with different truncation degrees are found again. The threshold:

```
> h3;
```

$$\mathrm{FAIL}$$

$k = 2$ does not exceed the threshold.

Apply the procedure again with $k = 5$:

```
> LaurentSolution(eq3, y(x), 'top' = 5, 'threshold' = 'h3');
```

$$\left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x),\ \_c_2 + \frac{1}{3}x\_c_2 + \frac{5}{6}x^2\_c_2 + \frac{13}{30}x^3\_c_2 + O\left(x^4\right)\right]$$

The threshold:

```
> h3;
```

$$3$$

The threshold is achieved in the computed truncated solutions with the valuation 0.

The equation is a prolongation of eq3:

```
> eq4:=(-1+x+x^2+9*x^3+O(x^4))*theta(y(x), x, 2)+
>       (-2+(x^3)/2+O(x^4))*theta(y(x), x, 1)+(x+6*x^2)*y(x);
```

$$eq4 := \left(-1 + x + x^2 + 9x^3 + O\left(x^4\right)\right)\theta(y(x), x, 2)+$$

$$\left(-2 + \frac{1}{2}x^3 + O\left(x^4\right)\right)\theta(y(x), x, 1) + \left(x + 6x^2\right)y(x)$$

```
> LaurentSolution(eq4, y(x), 'top' = 5, 'threshold' = 'h4');
```

$$\left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + \frac{1}{3}x\_c_2 + O\left(x^2\right), \_c_2 + \frac{1}{3}x\_c_2 + \frac{5}{6}x^2\_c_2 + \frac{13}{30}x^3\_c_2 + \frac{95}{144}x^4\_c_2 + O\left(x^5\right)\right]$$

The solutions are the prolongations of the solutions of eq3. The threshold:

```
> h4;
```

$$4$$

The threshold is achieved in the solution with valuation 0.