Construction of Resolving Sequences for Systems given by Ore Polynomials

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– S. Abramov, M. Petkovšek, A. Ryabenko, Hypergeometric solutions of first-order linear difference systems with rational-function coefficients, 18-th worshop on computer algebra, Dubna 2015, and CASC'2015, LNCS 9301, 2015, pp. 1–14.

– S. Abramov, M. Petkovšek, A. Ryabenko, Resolving sequences of operators for linear ordinary differential and difference systems, Computational Mathematics and Mathematical Physics (ЖВМиМФ) 56, 2016, pp. 894–910.

For differential systems with rational-function coefficients to find all formal exponential-logarithmic solutions and for difference systems with rational-function coefficients to find all hypergeometric solutions, the notion *resolving sequence of operators (equations)* was introduced.

For a difference or a differential system

$$A_n(x)y(x+n) + \cdots + A_1(x)y(x+1) + A_0(x)y(x) = 0,$$

$$A_n(x)y^{(n)}(x) + \cdots + A_1(x)y'(x) + A_0(x)y(x) = 0,$$

where $A_k \in \mathrm{Mat}_m(K(x))$, $k = 0, 1, \ldots, n$, $m \in \mathbb{N}_{>0}$, $A_n \neq 0$, K is a field of characteristic 0, $y(x) = (y_1(x), \ldots, y_m(x))^T$ is a vector-column of unknown functions, a resolving sequence of equations is

$$L_1 y_{\ell_1}(x) = 0, \ldots, \; L_j y_{\ell_j}(x) = 0, \ldots, \; L_p y_{\ell_p}(x) = 0,$$

where

$L_1, \ldots, L_p \in K(x)[\xi]$ is the resolving sequence of operators,
$\xi y(x) = y(x+1)$ or $\xi = \frac{d}{dx}$;
$\ell_1, \ldots, \ell_p \in \{1, 2, \ldots, m\}$, $\ell_i \neq \ell_j$ for $i \neq j$, is the indicator.

O. Ore, Theory of non-commutative polynomials, Annals of Mathematics 34, 1933, pp. 480–508.

Let $\mathbb{K}$ be a field of characteristic 0,

$\sigma : \mathbb{K} \mapsto \mathbb{K}$ be an automorphism:

$$\begin{array}{lcl} \sigma(a+b) & = & \sigma(a)+\sigma(b) \\ \sigma(ab) & = & \sigma(a)\sigma(b) \end{array} \quad \forall a, b \in \mathbb{K},$$

$\delta : \mathbb{K} \mapsto \mathbb{K}$ be a derivation with respect to σ:

$$\begin{array}{lcl} \delta(a+b) & = & \delta(a)+\delta(b) \\ \delta(ab) & = & \sigma(a)\delta(b)+\delta(a)b \end{array} \quad \forall a, b \in \mathbb{K}.$$

Let Z be an indeterminate.

An Ore polynomial ring (denote $\mathbb{K}[Z;\sigma,\delta]$) is a ring of

$$a_n Z^n + ... + a_1 Z + a_0,$$

where $a_n, \dots, a_1, a_0 \in \mathbb{K}$, with the usual polynomial addition, and the multiplication given by

$$Za = \sigma(a)Z + \delta(a), \quad \forall a \in \mathbb{K}.$$

– M. Bronstein, M. Petkovšek. On Ore rings, linear operators and factorisation, Programming and Computer Software (Программирование), No. 1, 1994, pp. 27–43.
– M. Bronstein, M. Petkošek. An introduction to pseudo-linear algebra, Theoretical Computer Science 157, 1996, pp. 3–33.

Let $\mathcal{L}_{\mathbb{K}}$ be a linear space over $\mathbb{K}$,
$\xi : \mathcal{L}_{\mathbb{K}} \mapsto \mathcal{L}_{\mathbb{K}}$ be a pseudo-linear map with respect σ and δ:

$$\begin{array}{l} \xi(u+v) = \xi(u) + \xi(v) \\ \xi(au) = \sigma(a)\,\xi(u) + \delta(a)\,u \end{array} \qquad \forall a \in \mathbb{K}, \forall u, v \in \mathcal{L}_{\mathbb{K}}.$$

$\mathbb{K} = K(x)$, $a(x) \in K(x)$, $y(x) \in \mathcal{L}_{K(x)}$

Derivation

$$\xi(y(x)) = \tfrac{d}{dx}y(x),\ \sigma(a(x)) = a(x),\ \delta(a(x)) = \tfrac{d}{dx}a(x)$$

Euler derivation

$$\xi(y(x)) = x\tfrac{d}{dx}y(x),\ \sigma(a(x)) = a(x),\ \delta(a(x)) = x\tfrac{d}{dx}a(x)$$

Shift

$$\xi(y(x)) = y(x+1),\ \sigma(a(x)) = a(x+1),\ \delta(a(x)) = 0$$

Difference

$$\xi(y(x)) = y(x+1) - y(x),\ \sigma(a(x)) = a(x+1),\ \delta(a(x)) = a(x+1) - a(x)$$

q-Shift

$$\xi(y(x)) = y(q\,x),\ \sigma(a(x)) = a(q\,x),\ \delta(a(x)) = 0$$

We consider a system

$$A_n\,\xi^n(y) + \cdots + A_1\,\xi(y) + A_0\,y = 0,$$

where

$A_k \in \mathrm{Mat}_m(\mathbb{K})$, $k = 0, 1, \ldots, n$, $m \in \mathbb{N}_{>0}$, $A_n \neq 0$,
$\mathbb{K}$ is a field of characteristic 0 with σ and δ,
$y = (y_1, \ldots, y_m)^T$ is a vector-column of unknown functions,
$\xi : \mathcal{L}_{\mathbb{K}} \mapsto \mathcal{L}_{\mathbb{K}}$ is a pseudo-linear map over $\mathbb{K}$.

The correspondent Ore polynomial

$$A_n\,Z^n + \cdots + A_1\,Z + A_0 \in \mathrm{Mat}_m(\mathbb{K})[Z; \sigma, \delta].$$

$$\xi(y) = Ay, \quad A \in \mathrm{Mat}_m(\mathbb{K})$$

1. Let $c^{[0]} = (\underbrace{0, \dots, 0}_{\ell_1 - 1}, 1, \underbrace{0, \dots, 0}_{m - \ell_1})$.
2. Compute

$$\begin{array}{c} c^{[1]} = \sigma(c^{[0]})\,A + \delta(c^{[0]}) \\ c^{[2]} = \sigma(c^{[1]})\,A + \delta(c^{[1]}) \\ \dots \\ c^{[k]} = \sigma(c^{[k-1]})\,A + \delta(c^{[k-1]}) \end{array} \qquad B = \begin{pmatrix} c^{[0]} \\ c^{[1]} \\ \vdots \\ c^{[k-1]} \end{pmatrix}$$

k is the least integer $(1 \leq k \leq m)$ such that $c^{[0]}, c^{[1]}, \dots, c^{[k]}$ are linearly dependent over $\mathbb{K}$:

$$u_k c^{[k]} + \dots + u_1 c^{[1]} + u_0 c^{[0]} = 0.$$

The equation

$$u_k \, \xi^k(y_{\ell_1}) + \cdots + u_1 \, \xi(y_{\ell_1}) + u_0 \, y_{\ell_1} = 0$$

is called the y_{ℓ_1}-*resolving equation*, the $(k \times m)$-matrix B is called the y_{ℓ_1}-*resolving matrix*.

$\boxed{k = m}$

If $\overline{y} \in \mathcal{L}_{\mathbb{K}}$ is a solution of y_{ℓ_1}-resolving equation

$$u_m \, \xi^m(y_{\ell_1}) + \cdots + u_1 \, \xi(y_{\ell_1}) + u_0 \, y_{\ell_1} = 0$$

then a solution of $\xi(y) = A \, y$ can be found by the system

$$B \, y = \begin{pmatrix} \overline{y} \\ \xi(\overline{y}) \\ \vdots \\ \xi^{m-1}(\overline{y}) \end{pmatrix}.$$

$\boxed{k < m}$

By y_{ℓ_1}-resolving equation, we can find all solutions of

$$\xi(y) = A\,y \quad \text{such that } y_{\ell_1} \neq 0.$$

For $y_{\ell_1} = 0$, we have

$$B\,y = 0.$$

There exist $m-k$ entries $y_{i_1}, \dots, y_{i_{m-k}}$ such that the other k entries can be expressed as linear forms in them.

The vector $\tilde{y} = (y_{i_1}, \dots, y_{i_{m-k}})^T$ satisfies

$$\xi\,\tilde{y} = \tilde{A}\,\tilde{y},$$

where $\tilde{A}$ is an $(m-k) \times (m-k)$-matrix.

For

$$\xi(y) = A\,y,$$

where $A \in \mathrm{Mat}_m(\mathbb{K})$,
$y = (y_1, \ldots, y_m)^T$,

we compute a resolving sequence of equations

$$L_1 y_{\ell_1} = 0, \ldots,\ L_j y_{\ell_j} = 0, \ldots,\ L_p y_{\ell_p} = 0,$$

where

$\ell_1, \ldots, \ell_p \in \{1, 2, \ldots, m\}$, $\ell_i \neq \ell_j$ for $i \neq j$, is the indicator;
$L_1, \ldots, L_p \in \mathbb{K}[\xi]$ is the resolving sequence of operators.

y(x+1)=A.y(x)

$$> A := \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)\,x} & \frac{x^2+x+1}{x} \end{bmatrix}:$$

http://www.ccas.ru/ca/resolvingsequence

```
> read "resolvingsequence.mpl";
> RS:-ResolvingSequence(A, y(x), OreTools:-SetOreRing(x,'shift'));
```

$$\Big[\big(-x^6 - x^5 + 4x^4 + 3x^3 - 3x^2 - 2x\big)\, y_1(x) + \big(2x^6 + 6x^5 + 2x^4 - 6x^3 - 6x^2 - 4x\big)\, y_1(x+1) + \big(-x^6 - 6x^5 - 11x^4 - 2x^3 + 8x^2 + 8x + 4\big)\, y_1(x+2) + \big(x^5 + 4x^4 + x^3 - 6x^2\big)\, y_1(x+3) = 0, \; -x\, y_4(x) + y_4(x+1) = 0\Big] \quad (1)$$

```
> LREtools:-hypergeomsols(-x y[4](x) + y[4](x+1) = 0, y[4](x), { }, output = basis)
```

$$[\Gamma(x)] \quad (2)$$

http://www.ccas.ru/ca/lrs

```
> read "lrshypergeomsols.mpl";
> LRS:-HypergeometricSolution(A, x);
```

$$\left[\begin{bmatrix} 0 \\ -\Gamma(x) \\ 0 \\ \Gamma(x) \end{bmatrix}\right] \quad (3)$$

For

$$A_n\,\xi^n(y) + \cdots + A_1\,\xi(y) + A_0\,y = 0,$$

the EG-elimination is used to get an embracing system

$$\overline{A}_n\,\xi^n(y) + \cdots + \overline{A}_1\,\xi(y) + \overline{A}_0\,y = 0,$$

where $\det \overline{A}_n \neq 0$, which is converted to an equivalent normal system

$$\xi(Y) = A\,Y,$$

where A is an $(mn \times mn)$-matrix,

$$Y = (y_1, \ldots, y_m, \xi y_1, \ldots, \xi y_m, \ldots, \xi^{n-1} y_1, \ldots, \xi^{n-1} y_m)^T.$$

Pre-defined

```
> DiffRing := OreTools:-SetOreRing(x,'differential');
```

$$DiffRing := UnivariateOreRing(x, differential) \quad (1)$$

```
> ShiftRing := OreTools:-SetOreRing(x,'shift');
```

$$ShiftRing := UnivariateOreRing(x, shift) \quad (2)$$

```
> QShiftRing := OreTools:-SetOreRing([x, q],'qshift');
```

$$QShiftRing := UnivariateOreRing(x, qshift) \quad (3)$$

```
> OreTools:-Properties:-Getdelta(DiffRing)
```

$$(p, var) \rightarrow \frac{\partial}{\partial var} p \quad (4)$$

User-defined

```
> DeltaRing := OreTools:-SetOreRing(x, 'Delta',
    'sigma' = ((p, x) → eval(p, x = x + 1)),
    'sigma_inverse' = ((p, x) → eval(p, x = x−1)),
    'delta' = ((p, x) → eval(p, x = x + 1) − p),
   'theta1' = 0);
```

$$DeltaRing := UnivariateOreRing(x, \Delta) \quad (5)$$

```
> OreTools:-Properties:-Getdelta(DeltaRing)
```

$$(p, x) \rightarrow p\Big|_{x = x + 1} - p \quad (6)$$

```
>
```

```
> DeltaRing := OreTools:-SetOreRing(x, 'Delta',
    'sigma' = ((p, x) -> eval(p, x = x + 1)),
    'sigma_inverse' = ((p, x) -> eval(p, x = x-1)),
    'delta' = ((p, x) -> eval(p, x = x + 1) - p),
   'theta1' = 0) :
```

$$> \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} . \Delta^{(2)}(y(x)) + \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix} . \Delta(y(x)) + \begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix} . y(x) = 0 :$$

http://www.ccas.ru/ca/resolvingsequence

```
> read "resolvingsequence.mpl" :
```

$$> RS\text{:-}ResolvingSequence\left(OrePoly\left(\begin{bmatrix} -x & 2 & x+2 \\ 2x & -2 & -x-2 \\ x & 0 & -x-2 \end{bmatrix}, \begin{bmatrix} x+1 & x+1 & 0 \\ -2x-4 & -2x-4 & 0 \\ -x-3 & -x-3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -x-5 & -x-2 \\ 0 & 2x+10 & x+2 \\ 1 & x+5 & x+2 \end{bmatrix} \right), DeltaRing \right);$$

Indicator( );

$$[OrePoly(-x+1, 2x+3, -3, -2x-7, x+6), OrePoly(-1, 0, 1)]$$

$$[1, 3] \qquad (1)$$

$$> (x+6)\,\Delta^{(4)}(y_1(x)) + (-2x-7)\,\Delta^{(3)}(y_1(x)) - 3\,\Delta^{(2)}(y_1(x)) + (2x+3)\,\Delta(y_1(x)) + (-x+1)\,y_1(x) = 0 :$$

$$> \Delta^{(2)}(y_3(x)) - y_3(x) = 0 :$$

\>

The package *RS* is available on http://www.ccas.ru/ca/resolvingsequence

The main procedure is *ResolvingSequence*. It's arguments:

- a system of linear homogeneous equations with rational-function coefficients;
- a *UnivariateOreRing*-structure.

The system can be represented by *OrePoly*-structure or (in the differential, difference or *q*-difference cases) in an explicit form.

Additional procedures are *Indicator*, *ResolvingEquation*, *ResolvingMatrix*, *EG*, *ResolvingDependence*, *CyclicVector*.

$$> \begin{bmatrix} x^5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \left(\frac{d^2}{dx^2} y(x) \right) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left(\frac{d}{dx} y(x) \right) + \begin{bmatrix} 0 & -x+2 \\ x+2 & 0 \end{bmatrix} \cdot y(x) = 0 :$$

$$> syst := \left\{ x^5 \left(\frac{d^2}{dx^2} y1(x) \right) + (-x+2)\, y2(x) = 0, \frac{d}{dx} y2(x) + (x+2)\, y1(x) = 0 \right\} :$$

http://www.ccas.ru/ca/resolvingsequence

```
> read "resolvingsequence.mpl";
> L := RS:-ResolvingSequence(syst, {y1(x), y2(x)}, OreTools:-SetOreRing(x,'differential'));
```

$$L := \left[(x^5 - 2x^4 - 8x^3 + 16x^2 + 16x - 32) \left(\frac{d}{dx} y1(x) \right) + (8x^6 - 18x^5 - 60x^4 + 160x^3) \left(\frac{d^2}{dx^2} y1(x) \right) + (7x^7 - 16x^6 - 36x^5 + 80x^4) \left(\frac{d^3}{dx^3} y1(x) \right) + (x^8 - 2x^7 - 4x^6 + 8x^5) \left(\frac{d^4}{dx^4} y1(x) \right) = 0 \right] \quad (1)$$

```
> DEtools:-formal_sol(L[1], y1(x), t,'order'= 1);
```

$$\left[[1 + O(t), t = x], \left[e^{-\frac{6}{t^2}} (1 + O(t)), \frac{1}{4} t^3 = x \right] \right] \quad (2)$$

http://www.ccas.ru/ca/formalsolution

```
> read "ldsformalsols.mpl";
> FormalSolution(syst, [y1(x), y2(x)], t, 'order'= 2);
```

$$\left[e^{-\frac{6}{t^2}} \cdot \begin{bmatrix} -8 + O(t^2) \\ t^5 + O(t^7) \end{bmatrix}, \frac{1}{4} t^3 = x \right] \quad (3)$$

```
> restart;
y(x+1)=A.y(x)
```

$$> A := \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)\,x} & \frac{x^2+x+1}{x} \end{bmatrix}:$$

```
> read "resolvingsequence.mpl";
> L := RS:-ResolvingEquation(A, y(x), OreTools:-SetOreRing(x,'shift'));
```

$$L := \left(-x^6 - x^5 + 4\,x^4 + 3\,x^3 - 3\,x^2 - 2\,x\right) y_1(x) + \left(2\,x^6 + 6\,x^5 + 2\,x^4 - 6\,x^3 - 6\,x^2 - 4\,x\right) y_1(x+1) + \left(-x^6 - 6\,x^5 - 11\,x^4 - 2\,x^3 + 8\,x^2 + 8\,x + 4\right) y_1(x+2) + \left(x^5 + 4\,x^4 + x^3 - 6\,x^2\right) y_1(x+3) = 0 \qquad \textbf{(1)}$$

```
> RS:-ResolvingMatrix()[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2\,(x^2+x-1)}{(x+2)\,(x+1)} & -\frac{x}{x+2} & -\frac{x\,(x-1)\,(x^2+3\,x+3)}{(x+1)^2\,(x+2)} & -\frac{x}{x+2} \end{bmatrix} \qquad \textbf{(2)}$$

```
> LREtools:-hypergeomsols( L, y[1](x), { }, output = basis);
```

$$\left[\frac{1}{x-1}\right] \qquad \textbf{(3)}$$

$$> \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2\left(x^2+x-1\right)}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)\left(x^2+3x+3\right)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} . \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} y_1(x) \\ y_1(x+1) \\ y_1(x+2) \end{bmatrix} :$$

$$> \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2x^2+2x-2}{(x+2)(x+1)} & -\frac{x}{x+2} & -\frac{x(x-1)\left(x^2+3x+3\right)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{bmatrix} . \begin{bmatrix} \frac{1}{x-1} \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{x+1} \end{bmatrix} :$$

$$> \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{x-1} \\ -y_4(x) + \frac{1}{x-1} \\ 0 \\ y_4(x) \end{bmatrix} :$$

$$> \begin{bmatrix} y_1(x+1) \\ y_2(x+1) \\ y_3(x+1) \\ y_4(x+1) \end{bmatrix} = \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)\,x} & \frac{x^2+x+1}{x} \end{bmatrix} . \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} :$$

$$> \begin{bmatrix} \frac{1}{x} \\ y_2(x+1) \\ y_3(x+1) \\ y_4(x+1) \end{bmatrix} = \begin{bmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{(x+1)\,x} & \frac{x^2+x+1}{x} \end{bmatrix} . \begin{bmatrix} \frac{1}{x-1} \\ -y_4(x) + \frac{1}{x-1} \\ 0 \\ y_4(x) \end{bmatrix} :$$

$$> \begin{bmatrix} y_2(x+1) \\ y_3(x+1) \\ y_4(x+1) \end{bmatrix} = \begin{bmatrix} -x\,y_4(x) + \frac{1}{x-1} \\ 0 \\ x\,y_4(x) - \frac{1}{(x-1)\,x} \end{bmatrix} :$$

$$
> \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x-1} \\ \Gamma(x) \left(\displaystyle\sum_{x1=2}^{x-1} \frac{1}{x1\ (x1-1)\ \Gamma(x1+1)} \right) + \dfrac{1}{x-1} \\ 0 \\ -\Gamma(x) \left(\displaystyle\sum_{x1=2}^{x-1} \frac{1}{x1\ (x1-1)\ \Gamma(x1+1)} \right) \end{bmatrix}
$$