Optimisation of Computer Algebra Techniques Application for Rician Data Analysis

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Abstract

- The paper presents a mathematical research directed on the optimization of the computer-algebra methods' application for solving the task of stochastic data analysis. Within the conducted theoretical investigation *a few mathematical techniques of the statistical data analysis have been elaborated which allow essential simplifying of the task solution by computer algebra methods.* The developed two-parameter approach to data analysis is efficiently applicable to a wide spectrum of scientific and applied tasks, in which a signal to be analyzed is described by the Rice statistical model.

Optimization of computer algebra techniques' application for two-parameter analysis of stochastic data:

- Two-parameter Rician signals analysis provides joint computing of both the informative and the noise signal's components (in contrast to non-realistic one-parameter approximation) and is in demand in applications implying operation in a real time mode.

- *The subject of the research*: simplification (optimization) of the algorithms of the Rician parameters joint computing.

- The compute-intensive task of solving the system of two nonlinear equations with two variables was reduced to solving just one equation for one variable.

- The possibility was proved of joint computing of both the signal and noise parameters without additional computations if compared with one-parameter approximation.

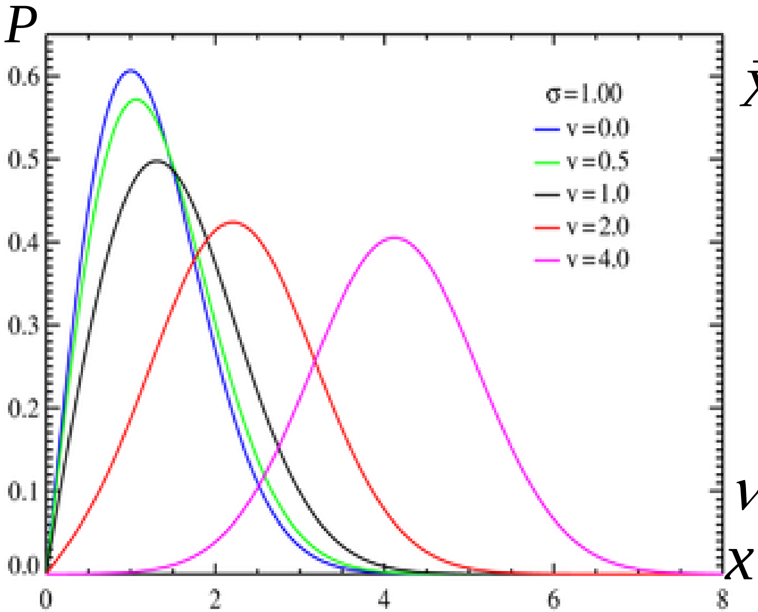
- The elaborated techniques may be applied in data processing systems with the priority of operation in a real time mode.

The Rice statistical distribution characterizes a value of an amplitude, or envelope of the complex signal composed as a sum of the sought-for initial signal and a Gaussian noise.

The fields of application:

- *magnetic-resonance visualization,*
- *radio signals reception and processing,*
- *radar signals analysis,*
- *analysis of the sonar signals,*
- *optical medium's properties measurements, etc.*

The Rice distribution properties

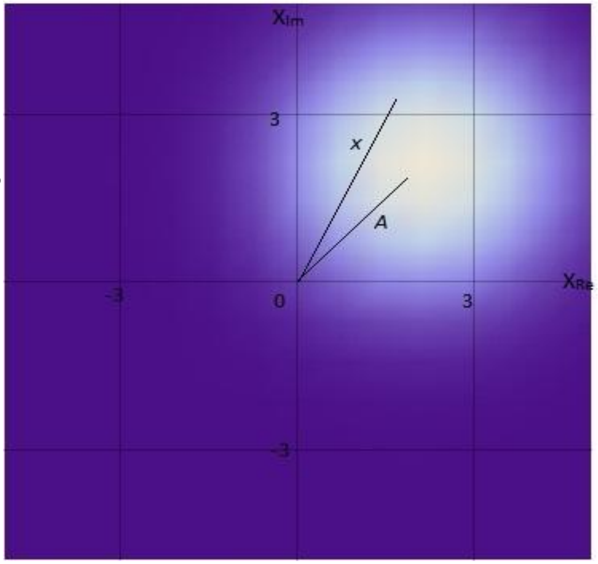


$$\vec{X}(x,\varphi)=\vec{A}+\vec{r} \qquad \vec{r}=\vec{r}(r_{\mathrm{Re}},r_{\mathrm{Im}})$$ - Gaussian noise

$$x=\sqrt{x_{\mathrm{Re}}^2+x_{\mathrm{Im}}^2} \qquad \overline{r_{\mathrm{Re}}{}^2}=\overline{r_{\mathrm{Im}}{}^2}=\sigma^2 \quad \overline{r}_{\mathrm{Re}}=\overline{r}_{\mathrm{Im}}=0$$

$$A=\nu$$

$\nu,\sigma^2 -$ Rician parameters



$$P\left(x\middle|\nu,\sigma^2\right)=\frac{x}{\sigma^2}\cdot\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$

$$\overline{x}=\sigma\cdot\sqrt{\pi/2}\cdot L_{1/2}\left(-\nu^2/2\sigma^2\right)$$

$$\sigma_x^2=\overline{x^2}-\overline{x}^2=2\sigma^2+\nu^2-\frac{\pi\sigma^2}{2}\cdot L_{1/2}^2\left(-\nu^2/2\sigma^2\right) \qquad L_{1/2}(z)=e^{z/2}\left[(1-z)I_0\left(\frac{-z}{2}\right)-zI_1\left(\frac{-z}{2}\right)\right]$$

Formulation of the problem of random Rician value two-parameter analysis

Initially determined complex value with amplitude ν is distorted by a Gaussian noise of dispersion σ^2,

$\Longrightarrow$

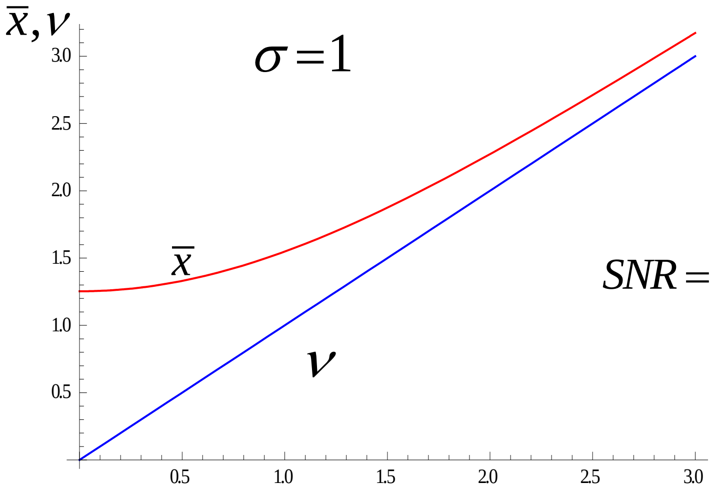
Amplitude $x = \sqrt{x_{\mathrm{Re}}{}^2 + x_{\mathrm{Im}}{}^2}$ of the resulting signal obeys the Rice statistical distribution with parameters ν and σ^2.

x_{Re}, x_{Im} – independent Gaussian values with dispersion σ^2

$$P\left(x \middle| \nu, \sigma^2\right) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$

The task consists in ***<u>reconstruction of useful (informative) signal</u>*** ν *against the noise background* ***<u>by joint computing the both Rician parameters</u>*** *on the basis of measurements of the summary signal* x_i ($i = 1, \ldots, n$), without any a priory assumptions concerning value of σ^2.

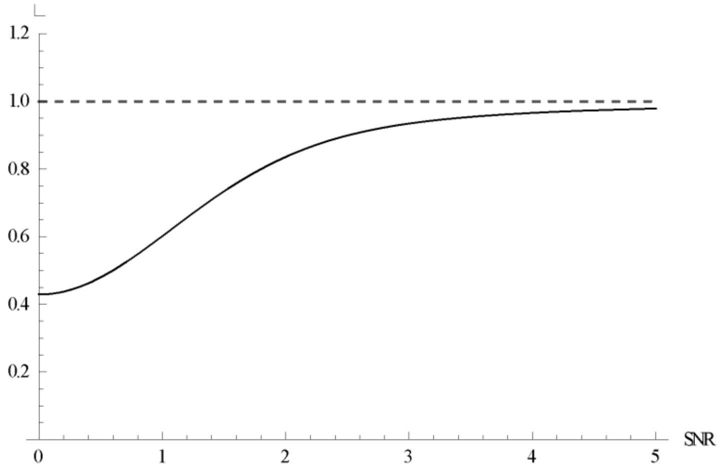
Nonlinear properties of the Rice distribution

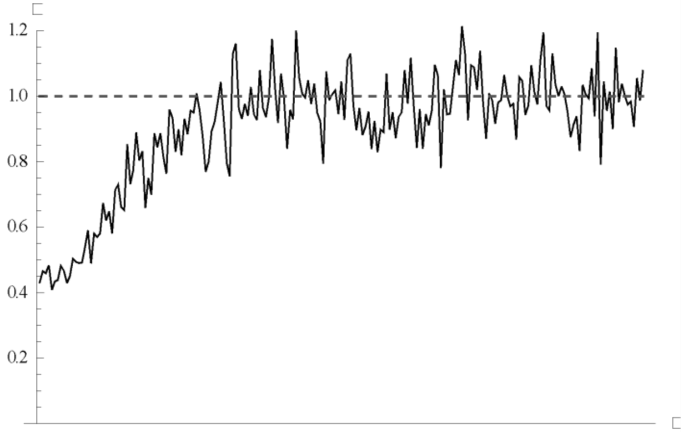


$SNR = \nu / \sigma$

$$\bar{x} = \sigma \cdot \sqrt{\pi / 2} \cdot L_{1/2}\left(-\nu^2 / 2\sigma^2\right)$$

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 2 \cdot \sigma^2 + \nu^2 - \sigma^2 \cdot \frac{\pi}{2} \cdot L_{1/2}^2\left(-\nu^2 / 2\sigma^2\right)$$





(a) - theory

(b) – numerical experiment

THE LIKELIHOOD FUNCTION OF THE RICE STATISTICAL DISTRIBUTION

ν - a sough-for amplitude of un-noised signal x ;

σ^2- a dispersion of the Gaussian noise distorting components x_{Re} and x_{Im} of signal $x = \sqrt{x_{\mathrm{Re}}^2 + x_{\mathrm{Im}}^2}$

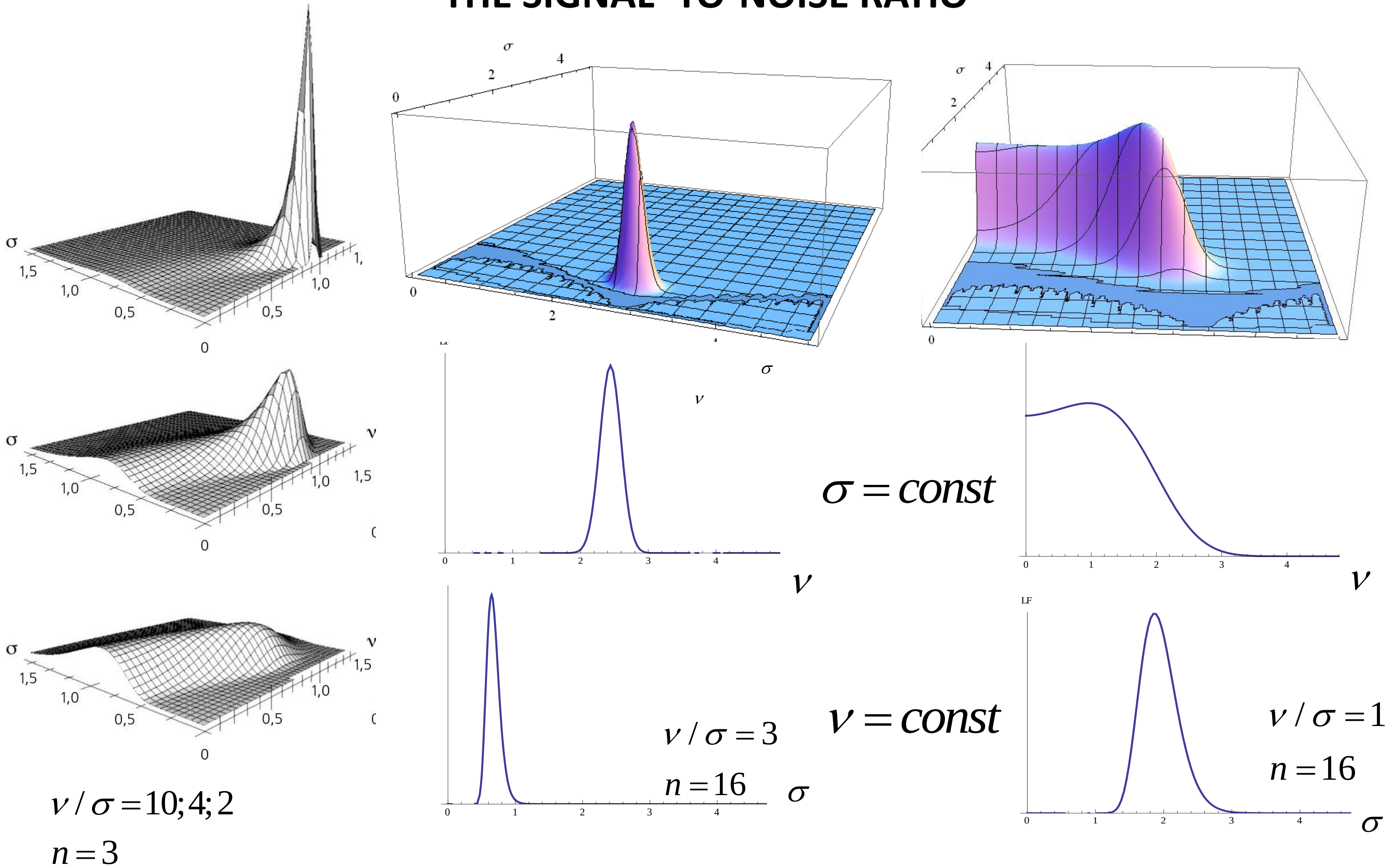
$$P\left(x \middle| \nu, \sigma^2\right) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2 + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$

The likelihood function (i.e. a joint probability density function of the events resulting in $x = x_i$ $(i = 1, \ldots, n)$, :

$$L\left(\vec{x} \middle| \nu, \sigma^2\right) = \prod_{i=1}^{n} P\left(x_i \middle| \nu, \sigma\right) = \prod_{i=1}^{n} \frac{x_i}{\sigma^2} \cdot \exp\left(-\frac{x_i^{\,2} + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{x_i \nu}{\sigma^2}\right)$$

$$\ln L\left(\vec{x} \middle| \nu, \sigma^2\right) = \sum_{i=1}^{n} \ln P\left(x_i \middle| \nu, \sigma\right) = \sum_{i=1}^{n} \left\{-2 \cdot \ln \sigma - \frac{x_i^2 + \nu^2}{2 \cdot \sigma^2} + \ln I_0\left(\frac{x_i \nu}{\sigma^2}\right)\right\}$$

RICIAN LIKELIHOOD FUNCTION AS DEPENDENT UPON THE SIGNAL–TO-NOISE RATIO



TWO-PARAMETER COMBINED ML-MM METHOD

$\nu, \sigma^2 -$ Rician parameters

$$\sum_{i=1}^{n} \frac{\partial}{\partial \nu} \ln I_0\left(\frac{x_i \nu}{\sigma^2}\right) - \frac{n \cdot \nu}{\sigma^2} = 0 \qquad \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i \cdot \frac{I_1\left(\frac{x_i \nu}{\sigma^2}\right)}{I_0\left(\frac{x_i \nu}{\sigma^2}\right)} - \frac{n \cdot \nu}{\sigma^2} = 0$$

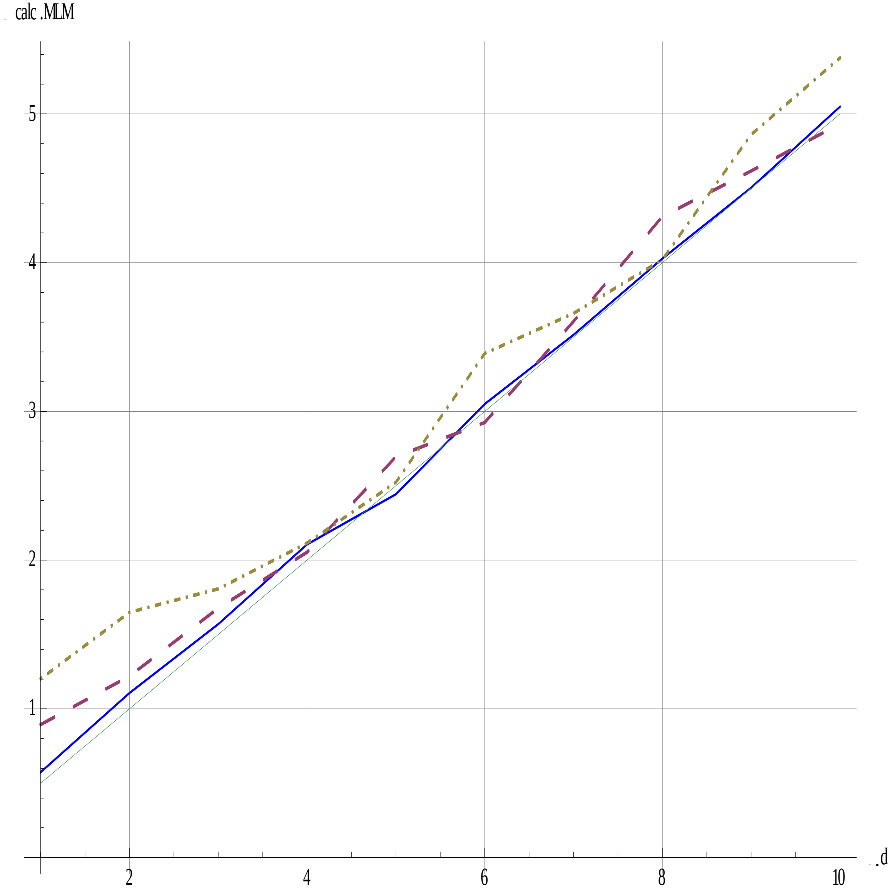
$$\bar{x} = \sigma \cdot \sqrt{\pi/2} \cdot L_{1/2}\left(-\nu^2/2\sigma^2\right) \qquad r = \frac{\nu^2}{2\sigma^2}$$

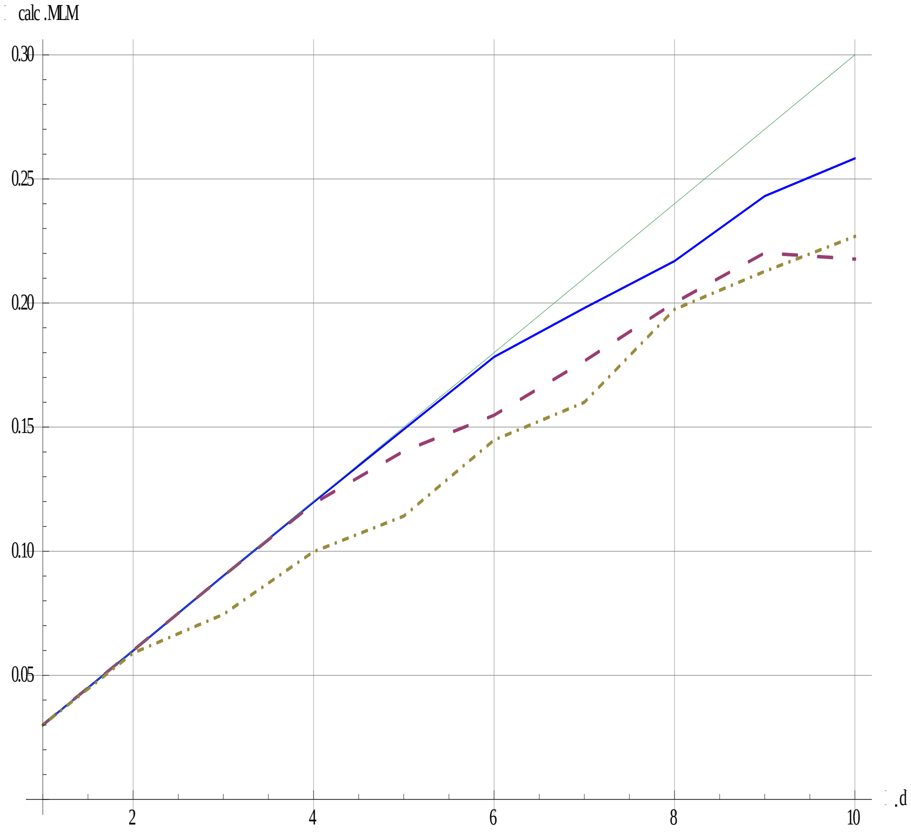
$$\begin{cases} \nu = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{x_i \cdot \nu}{\sigma^2}\right) \\ \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} x_i = \sigma \sqrt{\dfrac{\pi}{2}} \cdot L_{1/2}\left(-\dfrac{\nu^2}{2\sigma^2}\right) \end{cases} \Longrightarrow 2\sqrt{r/\pi} \cdot \frac{\langle x \rangle}{L_{1/2}(-r)} = \frac{1}{n} \sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i \sqrt{\pi \cdot r}}{\langle x \rangle} L_{1/2}(-r)\right)$$

$L_q(z)$ - the Laguerre polynomial

$$\sigma = \frac{\langle x \rangle}{\sqrt{\pi/2} \cdot L_{1/2}(-r)} \qquad \nu = \sqrt{2\sigma^2 r}$$

Computer simulation results for ML-MM method





σ= 0,5 (solid line); 1 (dashed line); 1,5 (dotted-dashed line)

ν=3.0 (solid line); 2.0 (dashed line); 1.0 (dotted-dashed line)

n=4

MAXIMUM LIKELIHOOD (ML) TECHNIQUE FOR SOLVING THE TASK OF TWO-PARAMETER RICIAN DATA ANALYSIS

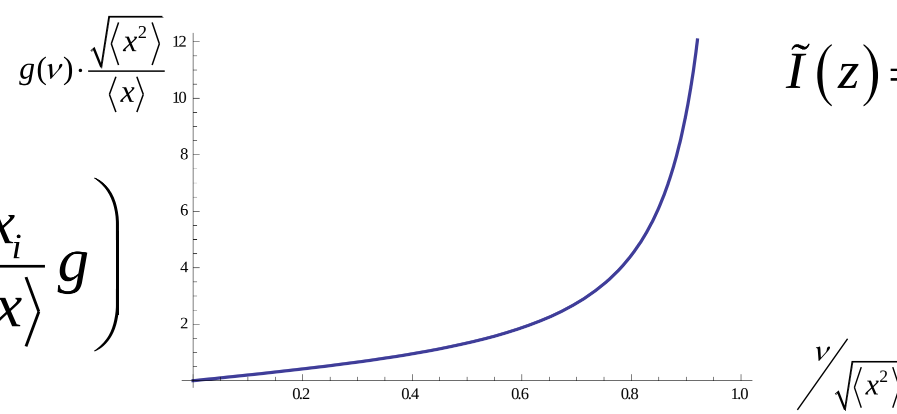
The key ML equations' system for parameters ν and σ^2:

$$\begin{cases} \dfrac{\partial}{\partial \nu} \ln L\left(\vec{x} \middle| \nu, \sigma^2\right) = 0 \\ \dfrac{\partial}{\partial \sigma} \ln L\left(\vec{x} \middle| \nu, \sigma^2\right) = 0 \end{cases} \Rightarrow \begin{cases} \nu = \dfrac{1}{n} \sum\limits_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{x_i \nu}{\sigma^2}\right) \\ \sigma^2 = \dfrac{1}{2 \cdot n} \sum\limits_{i=1}^{n} \left(x_i^2 + \nu^2\right) - \dfrac{\nu}{n} \sum\limits_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{x_i \nu}{\sigma^2}\right) \end{cases} \quad \begin{cases} \nu = \dfrac{1}{n} \sum\limits_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{2 \cdot x_i \cdot \nu}{\left\langle x^2 \right\rangle - \nu^2}\right) \\ \sigma^2 = \dfrac{1}{2} \cdot \left(\left\langle x^2 \right\rangle - \nu^2\right) \end{cases}$$

THEOREM: ***<u>Solution of the ML equation' system exists and is a unique one</u>***

$$g(\nu) = \frac{2 \cdot \langle x \rangle \cdot \nu}{\left\langle x^2 \right\rangle - \nu^2}$$

$$\nu = \xi\left(g(\nu)\right) \quad \xi(g) = \frac{1}{n} \sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i}{\langle x \rangle} g\right)$$



$$\tilde{I}(z) = \frac{I_1(z)}{I_0(z)}$$

Variants of the Method of Moments at solving the Task of Two-parameter Analysis of Rician Data

1. The method of lower even-numbered moments (MM24);

2. The method of lower moments (MM12).

Two-parameter method MM24

$$\begin{cases} \overline{x^2} = 2\cdot\sigma^2 + \nu^2 \\ \overline{x^4} = 8\cdot\sigma^4 + 8\cdot\sigma^2\cdot\nu^2 + \nu^4 \end{cases} \Longrightarrow \quad \begin{aligned} \nu^2 &= \langle x^2\rangle\sqrt{1-t} \\ \sigma^2 &= \frac{\langle x^2\rangle}{2}\left(1-\sqrt{1-t}\right) \end{aligned}$$

$$t = \frac{\langle x^4\rangle}{\left(\langle x^2\rangle\right)^2} - 1 \qquad 0 < t \le 1$$

At $\nu = 0 \quad t = 1 \qquad \sigma^2 = \frac{\langle x^2\rangle}{2}$

Two-parameter method MM12

1-st moment of rician value: $$\bar{x} = \sigma \cdot \sqrt{\pi/2} \cdot L_{1/2}\left(-\nu^2/2\sigma^2\right)$$

$L_{1/2}$ - Laguerre polynomial: $$L_{1/2}(z) = e^{z/2}\left[\left(1-z\right)I_0\left(\frac{-z}{2}\right) - zI_1\left(\frac{-z}{2}\right)\right]$$

$$\begin{cases} \sigma \cdot \sqrt{\pi/2} \cdot e^{-\frac{\nu^2}{4\sigma^2}}\left[\left(1+\frac{\nu^2}{2\sigma^2}\right)I_0\left(\frac{\nu^2}{4\sigma^2}\right) + \frac{\nu^2}{2\sigma^2}I_1\left(\frac{\nu^2}{4\sigma^2}\right)\right] = \bar{x} \\ 2\sigma^2 + \nu^2 = \overline{x^2} \end{cases}$$

$$\begin{cases} \sqrt{\frac{\pi}{2} \cdot \sigma^2} \cdot e^{-\frac{r}{2}} \cdot \left[\left(1+r\right)I_0\left(\frac{r}{2}\right) + rI_1\left(\frac{r}{2}\right)\right] = \langle x \rangle \\ 2\sigma^2\left(1+r\right) = \langle x^2 \rangle \end{cases} \qquad r = \frac{\nu^2}{2\sigma^2}$$

MM12 – method: equation for r

$$\sqrt{\frac{\pi}{4} \cdot \langle x^2 \rangle} \cdot \sqrt{1+r} \cdot e^{-\frac{r}{2}} \cdot I_0\left(\frac{r}{2}\right)\left[1 + \frac{r}{(1+r)} \cdot \tilde{I}\left(\frac{r}{2}\right)\right] = \langle x \rangle$$

$$\nu = \sqrt{\frac{r}{1+r}}\sqrt{\langle x^2 \rangle}$$

$$\sigma^2 = \frac{\langle x^2 \rangle}{2\left(1+r\right)}$$

TWO-PARAMETER COMBINED ML-MM METHOD

$\nu, \sigma^2 -$ Rician parameters

$$\begin{cases} \nu = \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \tilde{I}\left(\frac{x_i \cdot \nu}{\sigma^2}\right) \\ \frac{1}{n}\sum_{i=1}^{n} x_i = \sigma\sqrt{\frac{\pi}{2}} \cdot L_{1/2}\left(-\frac{\nu^2}{2\sigma^2}\right) \end{cases} \Longrightarrow \quad 2\sqrt{r/\pi} \cdot \frac{\langle x \rangle}{L_{1/2}(-r)} = \frac{1}{n}\sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i\sqrt{\pi \cdot r}}{\langle x \rangle} L_{1/2}(-r)\right)$$

$L_q(z)$ - the Laguerre polynomial

$$r = \frac{\nu^2}{2\sigma^2} \qquad \nu = \sqrt{2\sigma^2 r}$$

TWO-PARAMETER METHOD OF MOMENTS MM13

$$\overline{x} = \sigma \cdot \sqrt{\pi/2} \cdot L_{1/2}\left(-\nu^2/2\sigma^2\right),$$
$$\overline{x^3} = 3 \cdot \sigma^3 \cdot \sqrt{\pi/2} \cdot L_{3/2}\left(-\nu^2/2\sigma^2\right).$$

$$\Longrightarrow \langle x \rangle^3 \cdot {}_1F_1\left(-\frac{3}{2};1;-r\right) = \langle x^3 \rangle \cdot \frac{\pi}{6} \cdot {}_1F_1^3\left(-\frac{1}{2};1;-r\right),$$

$$\sigma = \langle x \rangle \sqrt{\frac{2}{\pi}} \Big/ {}_1F_1\left(-\frac{1}{2};1;-r\right)$$

${}_1F_1$ -confluent hypergeometric function of the 1-st order, or Kummer's function

Conclusion

- ✓ The mathematical methods have been developed *to optimize the computer algebra techniques for solving the two-parameter task of the both signal and noise parameters joint computing* at stochastic data analysis, namely:

- ✓ The compute-intensive task of *solving the system of two essentially nonlinear equations with two sought-for variables has been mathematically reduced to solving just one equation for one unknown variable;*

- ✓ The possibility of *computing the both sought-for parameters by computer algebra techniques without any additional calculative capacities if compared with the traditional one-parameter approximation* has been ensured;

- ➢ *Decreasing the needed calculative resources* for the task under consideration *allows applying the elaborated techniques in information technologies and data processing systems with priority of operation in a real-time mode.*

- ➢ The presented research has revealed that *improvement of the computer algebra's mathematical means for simplifying the symbolic computation is an important and a quite solvable task.*

Thank you very much for your attention