Optimisation of Computer Algebra Techniques Application for Rician Data Analysis

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Abstract

• The paper presents a mathematical research directed on the optimization of the computer-algebra methods’ application for solving the task of stochastic data analysis. Within the conducted theoretical investigation a few mathematical techniques of the statistical data analysis have been elaborated which allow essential simplifying of the task solution by computer algebra methods. The developed two-parameter approach to data analysis is efficiently applicable to a wide spectrum of scientific and applied tasks, in which a signal to be analyzed is described by the Rice statistical model.
Optimization of computer algebra techniques’ application for two-parameter analysis of stochastic data:

- Two-parameter Rician signals analysis provides joint computing of both the informative and the noise signal’s components (in contrast to non-realistic one-parameter approximation) and is in demand in applications implying operation in a real time mode.

- The subject of the research: simplification (optimization) of the algorithms of the Rician parameters joint computing.

- The compute-intensive task of solving the system of two nonlinear equations with two variables was reduced to solving just one equation for one variable.

- The possibility was proved of joint computing of both the signal and noise parameters without additional computations if compared with one-parameter approximation.

- The elaborated techniques may be applied in data processing systems with the priority of operation in a real time mode.
The Rice statistical distribution characterizes a value of an amplitude, or envelope of the complex signal composed as a sum of the sought-for initial signal and a Gaussian noise.

The fields of application:

- magnetic-resonance visualization,
- radio signals reception and processing,
- radar signals analysis,
- analysis of the sonar signals,
- optical medium’s properties measurements, etc.
The Rice distribution properties

\[ X(x, \varphi) = \bar{A} + \bar{r} \quad \bar{r} = \bar{r}(r_{\text{Re}}, r_{\text{Im}}) \] - Gaussian noise

\[ x = \sqrt{x_{\text{Re}}^2 + x_{\text{Im}}^2} \]

\[ r_{\text{Re}}^2 = r_{\text{Im}}^2 = \sigma^2 \quad r_{\text{Re}} = r_{\text{Im}} = 0 \]

\[ A = \nu \]

\( \nu, \sigma^2 \) – Rician parameters

\[ P(x|\nu, \sigma^2) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2 + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{\nu x}{\sigma^2}\right) \]

\[ \bar{x} = \sigma \cdot \sqrt{\pi / 2} \cdot L_{1/2}\left(-\nu^2 / 2\sigma^2\right) \]

\[ \sigma_x^2 = \bar{x}^2 - \bar{x}^2 = 2\sigma^2 + \nu^2 - \frac{\pi \sigma^2}{2} \cdot L_{1/2}\left(-\nu^2 / 2\sigma^2\right) \]

\[ L_{1/2}(z) = e^{z/2} \left[ (1-z) I_0\left(\frac{-z}{2}\right) - z I_1\left(\frac{-z}{2}\right) \right] \]

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Formulation of the problem of random Rician value 
two-parameter analysis

Initially determined complex value with amplitude \( V \) is 
distorted by a Gaussian noise of dispersion \( \sigma^2 \),

Amplitude \( x = \sqrt{x_{\text{Re}}^2 + x_{\text{Im}}^2} \) of the resulting signal 
obeyls the Rice statistical distribution with parameters \( V \) and \( \sigma^2 \).

\[ P(x | V, \sigma^2) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2 + v^2}{2\sigma^2}\right) \cdot I_0\left(\frac{xv}{\sigma^2}\right) \]

The task consists in reconstruction of useful (informative) signal \( V \) against 
the noise background by joint computing the both Rician parameters

on the basis of measurements of the summary signal \( x_i \ (i = 1, \ldots, n) \), without 
any a priory assumptions concerning value of \( \sigma^2 \).
Nonlinear properties of the Rice distribution

\[ \bar{x} = \sigma \cdot \sqrt{\pi / 2} \cdot L^{1/2} \left( -\nu^2 / 2\sigma^2 \right) \]

\[ \sigma_x^2 = \bar{x}^2 - \bar{x}^2 = 2 \cdot \sigma^2 + \nu^2 - \sigma^2 \cdot \frac{\pi}{2} \cdot L^2 \left( -\nu^2 / 2\sigma^2 \right) \]

(a) - theory

(b) – numerical experiment

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THE LIKELIHOOD FUNCTION OF THE RICE STATISTICAL DISTRIBUTION

\( \mathcal{N} \) - a sought-for amplitude of un-noised signal \( x \);  
\( \sigma^2 \) - a dispersion of the Gaussian noise distorting components \( x_{\text{Re}} \) and \( x_{\text{Im}} \) of signal \( x = \sqrt{x_{\text{Re}}^2 + x_{\text{Im}}^2} \)

\[
P(x|\nu, \sigma^2) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2 + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{\nu}{\sigma^2}\right)
\]

**The likelihood function** (i.e. a joint probability density function of the events resulting in \( x = x_i \) (\( i = 1, \ldots, n \))):

\[
L\left(\mathbf{x}|\nu, \sigma^2\right) = \prod_{i=1}^{n} P(x_i|\nu, \sigma) = \prod_{i=1}^{n} \frac{x_i}{\sigma^2} \cdot \exp\left(-\frac{x_i^2 + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{x_i\nu}{\sigma^2}\right)
\]

\[
\ln L\left(\mathbf{x}|\nu, \sigma^2\right) = \sum_{i=1}^{n} \ln P(x_i|\nu, \sigma) = \sum_{i=1}^{n} \left\{ -2 \cdot \ln \sigma - \frac{x_i^2 + \nu^2}{2 \cdot \sigma^2} + \ln I_0\left(\frac{x_i\nu}{\sigma^2}\right) \right\}
\]
RICIAN LIKELIHOOD FUNCTION AS DEPENDENT UPON THE SIGNAL–TO-NOISE RATIO

\[ \frac{n}{\sigma} = 3 \quad \frac{\nu}{\sigma} = 10; 4; 2 \quad n = 3 \]

\[ \sigma = const \]

\[ \nu = const \]

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TWO-PARAMETER COMBINED ML-MM METHOD

\[ \nu, \sigma^2 \text{ - Rician parameters} \]

\[ \sum_{i=1}^{n} \frac{\partial}{\partial \nu} \ln I_0 \left( \frac{x_i \nu}{\sigma^2} \right) - \frac{n \cdot \nu}{\sigma^2} = 0 \]

\[ \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i \cdot \frac{I_1 \left( \frac{x_i \nu}{\sigma^2} \right)}{I_0 \left( \frac{x_i \nu}{\sigma^2} \right)} - \frac{n \cdot \nu}{\sigma^2} = 0 \]

\[ \bar{x} = \sigma \cdot \sqrt{\frac{\pi}{2}} \cdot L_{1/2} \left( -\frac{\nu^2}{2\sigma^2} \right) \]

\[ r = \frac{\nu^2}{2\sigma^2} \]

\[ \begin{cases} \nu = \frac{1}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I} \left( \frac{x_i \cdot \nu}{\sigma^2} \right) \\ \frac{1}{n} \sum_{i=1}^{n} x_i = \sigma \sqrt{\frac{\pi}{2}} \cdot L_{1/2} \left( -\frac{\nu^2}{2\sigma^2} \right) \end{cases} \quad \Rightarrow 2\sqrt{r / \pi} \cdot \frac{\langle x \rangle}{L_{1/2} (-r)} = \frac{1}{n} \sum_{i=1}^{n} x_i \tilde{I} \left( \frac{x_i \sqrt{\pi \cdot r}}{\langle x \rangle} \right) L_{1/2} (-r) \]

\[ L_q (z) \text{ - the Laguerre polynomial} \]

\[ \sigma = \frac{\langle x \rangle}{\sqrt{\pi / 2} \cdot L_{1/2} (-r)} \quad \nu = \sqrt{2\sigma^2 r} \]
Computer simulation results for ML-MM method

\[ \phi = 0.5 \text{ (solid line); 1 (dashed line); 1.5 (dotted-dashed line)} \]

\[ \psi = 3.0 \text{ (solid line); 2.0 (dashed line); 1.0 (dotted-dashed line)} \]

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MAXIMUM LIKELIHOOD (ML) TECHNIQUE FOR SOLVING THE TASK OF TWO-PARAMETER RICIAN DATA ANALYSIS

The key ML equations’ system for parameters \( V \) and \( \sigma^2 \):

\[
\begin{align*}
\frac{\partial}{\partial V} \ln L\left( x | V, \sigma^2 \right) &= 0 \\
\frac{\partial}{\partial \sigma} \ln L\left( x | V, \sigma^2 \right) &= 0
\end{align*}
\]

\[
\begin{align*}
V &= \frac{1}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I}\left( \frac{x_i V}{\sigma^2} \right) \\
\sigma^2 &= \frac{1}{2 \cdot n} \sum_{i=1}^{n} \left( x_i^2 + V^2 \right) - \frac{V}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I}\left( \frac{x_i V}{\sigma^2} \right)
\end{align*}
\]

\[
\begin{align*}
V &= \frac{1}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I}\left( \frac{2 x_i \cdot V}{\langle x^2 \rangle - V^2} \right) \\
\sigma^2 &= \frac{1}{2} \cdot \left( \langle x^2 \rangle - V^2 \right)
\end{align*}
\]

**THEOREM:** Solution of the ML equation’ system exists and is a unique one

\[
\begin{align*}
g(V) &= \frac{2 \cdot \langle x \rangle \cdot V}{\langle x^2 \rangle - V^2} \\
g(V) \cdot \sqrt{\frac{\langle x^2 \rangle}{\langle x \rangle}} &= 12 \\
\tilde{I}(z) &= \frac{I_1(z)}{I_0(z)}
\end{align*}
\]

\[
\begin{align*}
V &= \xi(g(V)) \\
\xi(g) &= \frac{1}{n} \sum_{i=1}^{n} x_i \tilde{I}\left( \frac{x_i}{\langle x \rangle} \cdot g \right)
\end{align*}
\]
Variants of the Method of Moments at solving the Task of Two-parameter Analysis of Rician Data

1. The method of lower even-numbered moments (MM24);

2. The method of lower moments (MM12).

Two-parameter method MM24

\[
\begin{align*}
\bar{x}^2 &= 2 \cdot \sigma^2 + \nu^2 \\
\bar{x}^4 &= 8 \cdot \sigma^4 + 8 \cdot \sigma^2 \cdot \nu^2 + \nu^4
\end{align*}
\]

\[
t = \frac{\langle x^4 \rangle}{\left(\langle x^2 \rangle\right)^2} - 1 \quad 0 < t \leq 1
\]

\[
\nu^2 = \langle x^2 \rangle \sqrt{1-t}
\]

\[
\sigma^2 = \frac{\langle x^2 \rangle}{2} \left(1 - \sqrt{1-t}\right)
\]

At \( \nu = 0 \) \( t = 1 \) \( \sigma^2 = \frac{\langle x^2 \rangle}{2} \)
Two-parameter method MM12

1-st moment of rician value:

\[ \bar{x} = \sigma \cdot \sqrt{\pi / 2} \cdot L_{1/2} \left( -\nu^2 / 2\sigma^2 \right) \]

\( L_{1/2} \) - Laguerre polynomial:

\[
\begin{align*}
\left\{ \begin{array}{c}
\sigma \cdot \sqrt{\pi / 2} \cdot e^{-\frac{\nu^2}{4\sigma^2}} \left[ \left( 1 + \frac{\nu^2}{2\sigma^2} \right) I_0 \left( \frac{\nu^2}{4\sigma^2} \right) + \frac{\nu^2}{2\sigma^2} I_1 \left( \frac{\nu^2}{4\sigma^2} \right) \right] = \bar{x} \\
2\sigma^2 + \nu^2 = x^2 \\
\end{array} \right.
\]

\[
\begin{align*}
\left\{ \begin{array}{c}
\sqrt{\frac{\pi}{2}} \cdot \sigma \cdot e^{-\frac{r}{2}} \left[ \left( 1 + r \right) I_0 \left( \frac{r}{2} \right) + r I_1 \left( \frac{r}{2} \right) \right] = \langle x \rangle \\
2\sigma^2 \left( 1 + r \right) = \langle x^2 \rangle \\
\end{array} \right.
\]

MM12 – method: equation for \( r \)

\[
\sqrt{\frac{\pi}{4} \langle x^2 \rangle} \sqrt{1 + r} \cdot e^{-\frac{r}{2}} I_0 \left( \frac{r}{2} \right) \left[ 1 + \frac{r}{(1+r)} \cdot \tilde{I} \left( \frac{r}{2} \right) \right] = \langle x \rangle 
\]

\[
\nu = \sqrt{\frac{r}{1+r} \sqrt{\langle x^2 \rangle}} \\
\sigma^2 = \frac{\langle x^2 \rangle}{2(1+r)} 
\]
TWO-PARAMETER COMBINED ML-MM METHOD

\[ \nu, \sigma^2 \] - Rician parameters

\[
\begin{align*}
\nu &= \frac{1}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I} \left( \frac{x_i \cdot \nu}{\sigma^2} \right) \\
\frac{1}{n} \sum_{i=1}^{n} x_i &= \sigma \sqrt{\frac{\pi}{2}} \cdot L_{1/2} \left( \frac{-\nu^2}{2\sigma^2} \right)
\end{align*}
\]

\[ \Rightarrow 2\sqrt{r / \pi} \cdot \frac{\langle x \rangle}{L_{1/2}(-r)} = \frac{1}{n} \sum_{i=1}^{n} x_i \tilde{I} \left( \frac{x_i \sqrt{\pi \cdot r}}{\langle x \rangle} \right) L_{1/2}(-r) \]

\[ r = \frac{\nu^2}{2\sigma^2}, \quad \nu = \sqrt{2\sigma^2 r} \]

\[ L_q(z) \] - the Laguerre polynomial

TWO-PARAMETER METHOD OF MOMENTS MM13

\[ \bar{x} = \sigma \cdot \sqrt{\frac{\pi}{2}} \cdot L_{1/2} \left( -\frac{\nu^2}{2\sigma^2} \right), \]

\[ \bar{x}^3 = 3 \cdot \sigma^3 \cdot \sqrt{\frac{\pi}{2}} \cdot L_{3/2} \left( -\frac{\nu^2}{2\sigma^2} \right). \]

\[ \Rightarrow \langle x \rangle^3 \cdot \frac{1}{1} F_1 \left( \frac{-3}{2}; 1; -r \right) = \langle x^3 \rangle \cdot \frac{\pi}{6} \cdot \frac{1}{1} F_3 \left( \frac{-1}{2}; 1; -r \right), \]

\[ \sigma = \langle x \rangle \sqrt{\frac{2}{\pi}} / \frac{1}{1} F_1 \left( \frac{-1}{2}; 1; -r \right) \]

\[ \frac{1}{1} F_1 \] - confluent hypergeometric function of the 1-st order, or Kummer's function
Conclusion

✓ The mathematical methods have been developed to optimize the computer algebra techniques for solving the two-parameter task of the both signal and noise parameters joint computing at stochastic data analysis, namely:

✓ The compute-intensive task of solving the system of two essentially nonlinear equations with two sought-for variables has been mathematically reduced to solving just one equation for one unknown variable;

✓ The possibility of computing the both sought-for parameters by computer algebra techniques without any additional calculative capacities if compared with the traditional one-parameter approximation has been ensured;

➢ Decreasing the needed calculative resources for the task under consideration allows applying the elaborated techniques in information technologies and data processing systems with priority of operation in a real-time mode.

➢ The presented research has revealed that improvement of the computer algebra’s mathematical means for simplifying the symbolic computation is an important and a quite solvable task.

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Thank you very much for your attention