Algorithm EG as a Tool for Finding Laurent Solutions of Linear Differential Systems with Truncated Series Coefficients

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Starting Point

We consider systems of the form

$$A_r(x)\theta^r y(x) + A_{r-1}(x)\theta^{r-1} y(x) + \cdots + A_0(x)y(x) = 0 \qquad (1)$$

- $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is the vector of unknowns,
- $A_r(x), \ldots, A_0(x)$ are $m \times m$-matrices of coefficients whose entries are power series in x over the field of algebraic numbers,
- $\theta = x\frac{d}{dx}$.

Laurent solution of (1) is a solution $y(x) = (y_1(x), \ldots, y_m(x))^T$ the components of which are formal Laurent series:

$$y(x) = \sum_{n=v}^{\infty} u(n)x^n \qquad (2)$$

- $v \in \mathbb{Z}$ is a *valuation*
- $u(n) = (u_1(n), \ldots, u_m(n))^T$ are vectors of coefficients for $n \in \mathbb{Z}$.

The algorithm from

Abramov S.A., Barkatou M.A., Khmelnov D.E. On full rank differential systems with power series coefficients. J. Symbolic Comput. 2015. Vol. 68. P. 120–137.

finds all truncated Laurent solutions with any given truncation degree for a full-rank system with the algorithmically specified series (i.e. an algorithm is given which computes the coefficient of any term x^s of any series).
It is based on the construction of an *induced recurrent system*

$$B_0(n)u(n) + B_{-1}(n)u(n-1) + \ldots = 0 \tag{3}$$

- $u(n) = (u_1(n), \ldots, u_m(n))^T$ is the column vector of unknown sequences of solution coefficients
- $u_i(n) = 0$ for all negative n with large enough value of $|n|$, $i = 1, \ldots, m$
- $B_0(n), B_{-1}(n), \ldots$ are matrices of polynomials in n, $B_0(n)$ is the *leading matrix*.

It is constructed with the transformation $x \to E^{-1}$, $\theta \to n$, applied to the original differential system. E^{-1} is the shift operator: $E^{-1}u(n) = u(n-1)$.

- If $B_0(n)$ is non-singular, then we can consider $\det B_0(n) = 0$ as an indicial equation: the set of its integer roots includes the set of all possible valuations of the desired Laurent solutions. It makes possible to find the lower bound for the valuations of all Laurent solutions of the system.
- If $\det B_0(n) = 0$ has no integer roots, then the system has no Laurent solutions.
- If $B_0(n)$ is singular, then algorithm $\mathrm{EG}_\sigma^\infty$ (a special version of original EG algorithm for infinite recurrent systems) initially applied to transform the induced recurrent system to the embracing recurrent system of the same form with a non-singular leading matrix. Any given number of its initial terms is constructed. The embracing recurrent system supplemented with a set of linear constraints, which are also constructed by $\mathrm{EG}_\sigma^\infty$ algorithm, has the same set of solutions as the initial induced recurrent system.

In our current research we are focused on the case of *truncated systems* when the series represented in a truncated form $a(x) + O(x^{t+1})$, where $a(x)$ is a polynomial, the integer $t \geq \deg a(x)$ is a *truncation degree* (which might be different for different series).

The *prolongation* of a truncated series is a series (possibly, also truncated) the initial terms of which coincide with known initial terms of the original truncated series.

The prolongation of a truncated equation is an equation with the coefficients which are prolongations of coefficients of the original equation.

The prolongation of a truncated system is a system the equations of which are prolongations of the equations of the original system.

We are interested in finding the maximum possible number of terms of truncated Laurent solutions of a given truncated system that are invariant with respect to any prolongations of the truncated coefficients of the given system.

Solutions with arbitrary truncation degree cannot be calculated for a truncated system. For a particular case of a scalar equation ($m = 1$) it was proved in

Abramov S.A., Ryabenko A.A., Khmelnov D.E. Linear ordinary differential equations and truncated series. Comput. Math. and Math. Phys. 2019. Vol. 59, N. 10. P. 1649–1659.

We proposed there an algorithm which finds such truncated Laurent solutions for scalar equations. We utilized induced recurrent systems and *literals* as a foundation of the algorithm.

Literals are symbols used to represent unspecified coefficients of the truncated series involved in the system. We say that the coefficients of terms x^s with s greater than the truncation degree are *unspecified*.

Our algorithm for truncated systems is a modification of the algorithms for the systems with the algorithmically specified coefficients. The key idea is to represent the truncated series algorithmically: the algorithm returns the known coefficient of the series and it returns literals for the unspecified ones.

We applied the approach of the algorithm for the scalar case to the systems with $m > 1$ and proposed an algorithm for constructing Laurent solutions of the system for the case when the determinant of the leading matrix of the induced system is not zero (i.e. the leading matrix is non-singular) and does not contain literals in the work

Abramov S.A., Ryabenko A.A., Khmelnov D.E. Searching for Laurent solutions of systems of linear differential equations with truncated power series in the role of coefficients. Program. Comput. Software. 2023 (to be published).

Advances and Further Plans

Our advances in the research are related to the use of the algorithm $\mathrm{EG}_\sigma^\infty$ to extend the applicability of our algorithm to the systems whose induced recurrent systems have singular leading matrix. We continue the adaptation of the algorithm for the systems with the algorithmically specified coefficients with the help of literals.

The $\mathrm{EG}_\sigma^\infty$ algorithm consists in the successive repetition of reduction and shift steps, which continues until the rows of the leading matrix remain linear dependent. On the reduction step, coefficients of the dependence are found; then, the equation corresponding to one of the dependent rows is replaced with the linear combination of the other equations, hence the row of the leading matrix is set zero. On the shift step, the shift operator E is applied to the new equation.

The termination of the algorithm is guaranteed with using a simple rule when selecting equations to be replaced.

The reduction steps also lead to a finite set of linear constraints, each of which involves a finite number of elements of a sequential solution and is a linear combination of these elements with constant coefficients. The linear constraints correspond to the integer roots of a polynomial which is the coefficient of the row of the replaced equation in the linear combination (we further refer to the polynomials as the *constraint polynomials*).

The main obstacle is that literals may appear in intermediate calculations. If there are no literals both in the determinant of the leading matrix and in the constraint polynomials after $\mathrm{EG}_\sigma^\infty$ execution then the further calculations with the resulting embraced recurrent system and the linear constraints gain desired truncated Laurent solutions. It give us the new extended version of our algorithm in the case.

It is preliminary implemented in Maple as an updated version of procedure LaurentSolution of package TruncatedSeries (for more information about the package, please visit http://www.ccas.ru/ca/TruncatedSeries).

Let's consider the following system

$$\begin{pmatrix} 3x + O(x^2) & 7x^2 + O(x^4) \\ O(x^2) & 17x^2 + O(x^4) \end{pmatrix} \theta^2 y(x) + \\ + \begin{pmatrix} -1 + 2x + O(x^2) & x + 5x^2 + O(x^4) \\ O(x^2) & 11x^2 + O(x^4) \end{pmatrix} \theta y(x) + \\ + \begin{pmatrix} O(1) & x - 3x^2 + O(x^4) \\ 1 + O(x^2) & -6x^2 + O(x^4) \end{pmatrix} y(x) = 0. \quad (4)$$

The leading matrix of its induced recurrent system is singular:

$$\begin{pmatrix} U_{[1,1],[0,0]} - n & 0 \\ 1 & 0 \end{pmatrix}. \quad (5)$$

$U_{[i,j][k,s]}$ is the representation of the literal denoting an unspecified coefficient of x^s in (i,j)-th element of matrix coefficient $A_k(x)$ of the differential system.

After execution of $\mathrm{EG}_\sigma^\infty$ the transformed leading matrix becomes non-singular:

$$\begin{pmatrix} 3n^2+2n+U_{[1,1],[0,1]} & n+1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

and its determinant $(-n-1)$ contains no literals. No linear constraints constructed by $\mathrm{EG}_\sigma^\infty$, so there is no constraint polynomials with literals as well. It is the case when our new algorithm is applicable and it computes the truncated Laurent solution (c_1 is an arbitrary constant):

$$\begin{pmatrix} 6x^2c_1+O(x^3) \\ \frac{c_1}{x}+c_1+O(x) \end{pmatrix}. \tag{7}$$

Let's consider another system:

$$\begin{pmatrix} O(x^5) & -1+O(x^5) \\ 1+O(x^5) & O(x^5) \end{pmatrix} \theta y(x) + \begin{pmatrix} O(x^5) & O(1) \\ 2+O(x^5) & O(x^5) \end{pmatrix} y(x). \quad (8)$$

The leading matrix of its induced recurrent system is already non-singular:

$$\begin{pmatrix} 0 & U_{[1,2],[0,0]} - n \\ 2+n & 0 \end{pmatrix}. \quad (9)$$

However there is a literal in its determinant $(n - U_{[1,2],[0,0]})(2+n)$. It has the roots -2 and $U_{[1,2],[0,0]}$, so the set of integer roots of the determinant may be different for different integer values of the literal $U_{[1,2],[0,0]}$.

It is easy to check that there is no desired Laurent solutions of the system.

It may be done by constructing various prolongations of the original system with various integer values of the literal $U_{[1,2],[0,0]}$ and finding Laurent solutions of the prolongations with the previous version of our algorithm (is is applicable since for each of the prolongations the leading matrix of the induced recurrence is still non-singular and there are no literals in its determinant already).

For example, for $U_{[1,2],[0,0]} = 5$ the solution of the prolongation is

$$\begin{pmatrix} O(x^{10}) \\ c_1 x^5 + O(x^6) \end{pmatrix} \tag{10}$$

and for $U_{[1,2],[0,0]} = 6$ the solution of the prolongation is

$$\begin{pmatrix} O(x^{11}) \\ c_1 x^6 + O(x^7) \end{pmatrix}. \tag{11}$$

Since the solutions of the prolongations has no coinciding initial terms of the series, there is no desired Laurent solution of the original system.

Let $p(n)$ be the determinant of a leading matrix or be a constraint polynomial. If $p(n)$ contains literals then it may be represented as

$$p(n) = a(u_1, \dots, u_s)(n - r_1)\dots(n - r_k)(b_q(u_1, \dots, u_s)n^q + \dots + \\ + b_1(u_1, \dots, u_s)n + b_0(u_1, \dots, u_s)), \quad (12)$$

where $u_1, \dots, u_s$ are literals, $a(u_1, \dots, u_s)$, $b_0(u_1, \dots, u_s)$, $b_1(u_1, \dots, u_s), \cdots, b_q(u_1, \dots, u_s)$ are polynomials in the literals, $r_1 \dots r_k$ are integer roots of $p(n)$ independent from the literals.

- If $a(u_1, \dots, u_s)$ does contain literals (it is not a number) then there are such values of $u_1, \dots, u_s$ that $a(u_1, \dots, u_s) = 0$. Hence $p(n) = 0$, i.e. the leading matrix is singular for the values of the literals.
- If $a(u_1, \dots, u_s)$ is a number then the solution of the algebraic equation

  $$b_q(u_1, \dots, u_s)r_0^q + \dots + b_1(u_1, \dots, u_s)r_0 + b_0(u_1, \dots, u_s) = 0 \quad (13)$$

  in respect to $u_1, \dots, u_s$ gives such values of $u_1, \dots, u_s$ that $p(n)$ has any desired root r_0 in addition to $r_1 \dots r_k$.

Thus, in all cases the set of integer roots of $p(n)$ is not invariant in respect to the prolongations of the differential system in hand.

EG_σ^∞ may have variability in its execution. In spite of the fact that the special rule should be used for choosing the equation to be replaced, it is still possible to have more than one option to choose. So, EG_σ^∞ may result in different embraced systems for the same recurrent system.
E.g., for the first considered system the second variant of EG_σ^∞ execution gives the embraced recurrent system with the other leading matrix:

$$\begin{pmatrix} U_{[1,1],[0,0]} - n & 0 \\ 3n^2 + 2n + U_{[1,1],[0,1]} & n+1 \end{pmatrix}. \tag{14}$$

Its determinant $\left(U_{[1,1],[0,0]} - n\right)(n+1)$ contains the literal $U_{[1,1],[0,0]}$.

The determinant is very similar to the determinant of the leading matrix of the induced recurrence for the second considered system that has no desired Laurent solution. The second variant of EG_σ^∞ execution also gives the constraint polynomial $-U[[1,1],[0,0]] + n$ which contains the literal.

Still, we know from the first variant of EG_σ^∞ execution, that the system has desired Laurent solution. It gives us the counterexample to the conjecture that there is no desired Laurent solution as soon as the determinant of the leading matrix and/or the constraint polynomials contain literals.

We experiment with the modification of our algorithm for the case of determinants and constraint polynomials containing literals, which takes into account only invariant integer roots of the determinant and constraint polynomial (i.e. those roots which are independent of literals).

The experiments show that the modification of the algorithm gives correct answers for the first considered system for both variants of $\mathrm{EG}_\sigma^\infty$ execution and for the second considered system, as well as for more other systems. Our further plan are either to prove that the approach is always correct or to identify the limitations of its applicability.