Asymptotic Approximations and Symbolic Representation of Parametric Families of Feedback Controls in Nonlinear Systems

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Discrete nonlinear control system with a parameter

Let us consider the following nonlinear **stabilization** problem

$$x(t+1) = \varepsilon A(x)x(t) + B(x)u(t),$$

$$x(0) = x_0,\ x(t) \in X \subset R^n,\ u(t) \in R^r,\ t = 0,1,2...\ , \qquad (1)$$

$$\alpha_1 \le \varepsilon \le \alpha_2,$$

$$I(u) = \frac{1}{2}\sum_{t=0}^{\infty}(x^T Q(x,\varepsilon)x + u^T R_0 u) \to \min, \qquad (2)$$

where $X \subset R^n$ is some bounded set and $\alpha_1 \le \varepsilon \le \alpha_2,$

$\alpha_1, \alpha_2 -$ fixed positive numbers.

$R_0 > 0$ *and* $Q(x,\varepsilon)$ *is a positive definite matrix for*

$$x \in X,\ \ \alpha_1 \le \varepsilon \le \alpha_2.$$

Matrix discrete state dependent Riccati equation

We choose the control in the form of a formally linear feedback

$$u(x,\varepsilon) = -\varepsilon\left[R_0 + B(x(t))^T P(x(t)) B(x(t))\right]^{-1} B(x(t))^T P(x(t)) A(x(t)) x(t),$$

where matrices have state dependent coefficients and P is a solution for $x \in X$, $\alpha_1 \le \varepsilon \le \alpha_2$ of the matrix discrete state dependent Riccati equation (D-SDRE)

$$\varepsilon^2 A^T(x) P A(x) - P - \varepsilon^2 A^T(x) P B(x) \tilde{R}^{-1}(x) \times$$
$$\times B^T(x) P A(x) + Q(x,\varepsilon) = 0,$$

here $\tilde{R}(x) = (R_0 + B^T(x) P B(x))$ is an invertible matrix for all admissible $x \in X$.

It is assumed that the right-hand sides of the corresponding closed system obtained by the substitution of control are continuous and bounded in a convex domain.

Algorithm

Step 1. Construction of an asymptotic expansion of the matrix discrete state dependent Riccati equation solution in the right neighborhood of point $\alpha_1,\ \varepsilon = (\alpha_1 + \eta),\ \eta \to 0$

$$P_2^R(x,\eta) = P_0^R(x) + \eta P_1^R(x) + \eta^2 P_2^R(x)$$

$$Q(x,\eta) = Q_0(x) + (\alpha_1 + \eta) Q_1(x) + (\alpha_1 + \eta)^2 Q_2(x)$$

Step 2. Construction of an asymptotic expansion of the Riccati equation solution in the left neighborhood of point $\alpha_2,\ \varepsilon = \alpha_2 - \mu > 0,\ \mu \to \infty$

$$P_2^L(x,\mu) = P_0^L(x) + \mu P_1^L(x) + \mu^2 P_2^L(x)$$

$$Q(x,\mu) = Q_0(x) + (\alpha_2 - \mu) Q_1(x) + (\alpha_2 - \mu)^2 Q_2(x)$$

Step 3. These two approximations are combined into one symbolic construction using a two-point matrix Pade approximation of order [2/2].

Step 4. The obtained Pade approximation is used as the gain coefficient of the D-SDRE control.

Construction of the asymptotic expansion of the matrix discrete state dependent Riccati equation solution

Asymptotic expansion in the left neighborhood of α_2,

$$\boxed{P_2^L(x,\mu) = P_0^L(x) + \mu P_1^L(x) + \mu^2 P_2^L(x)} \qquad (3)$$

D-SDRE equation for $P_0^L(x)$

$$1)\ \alpha_2^{\,2} A^T(x) P_0^L(x) A(x) - P_0^L(x) -$$
$$-\alpha_2^{\,2} A^T(x) P_0^L(x) B(x) \tilde{R}^{-1}(x) B^T(x) P_0^L(x) A(x) + Q(x,0) = 0,$$

where

$$Q(x,0) = Q_0(x) + \alpha_2 Q_1(x) + \alpha_2^{\,2} Q_2(x),$$
$$\tilde{R}(x) = (R_0 + B^T(x) P_0^L(x) B(x))$$

2) Linear Lyapunov equation

$$(A_{cl}{}^{L})^T P_1^L A_{cl}{}^{L} - P_1^L + C_1{}^{L} = 0,$$
$$C_1{}^{L} = -Q_1(x) - 2\alpha_2 Q_2(x) - 3\alpha_2{}^2 Q_3(x) - 2\alpha_2 A^T(x) P_0^L A(x) +$$
$$+2\alpha_2 A^T(x) P_0^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_0^L A(x);$$
$$A_{cl}{}^{L} = \alpha_2 (A(x) - B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_0^L A(x))$$

3) Linear Lyapunov equation

$$(A_{cl}{}^{L})^T P_2^L A_{cl}{}^{L} - P_2^L + C_2{}^{L} = 0,$$
$$C_1{}^{L} = Q_2(x) + 3\alpha_2 Q_3(x) + A^T(x) P_0^L A(x) - 2A^T(x) P_1^L A(x)\alpha_2$$
$$-A^T(x) P_0^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_0^L A(x)$$
$$-A^T(x) P_1^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_1^L A(x)\alpha_2{}^2$$
$$+2\alpha_2 A^T(x) P_1^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_0^L A(x) + 2\alpha_2 A^T(x) P_0^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_1^L A(x) +$$
$$A^T(x) P_1^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_1^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_0^L A(x)\alpha_2{}^2$$
$$+A^T(x) P_0^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_1^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_1^L A(x)\alpha_2{}^2$$
$$-2A^T(x) P_0^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_1^L B(x)(R_0 + B^T(x) P_0^L B(x))^{-1} B^T(x) P_0^L A(x)\alpha_2;$$

The equation for $\boxed{P_2{}^{R}(x,\eta)}$ are found by analogy

Asymptotic expansion in the right neighborhood of α_1,

$$\boxed{P_2^R(x,\eta) = P_0^R(x) + \eta P_1^R(x) + \eta^2 P_2^R(x)}$$

1) $\alpha_1^2 A^T(x)P_0^R(x)A(x) - P_0^R(x) - \alpha_1^2 A^T(x)P_0^R(x)B(x)\tilde{R}^{-1}(x)B^T(x)P_0^R(x)A(x) + Q(x,0) = 0,$

$$Q(x,0) = Q_0(x) + \alpha_1 Q_1(x) + \alpha_1^2 Q_2(x),\ \tilde{R}(x) = (R_0 + B^T(x)P_0^R(x)B(x))$$

2)
$$\begin{aligned}
&(A_{cl}^R)^T P_1^R A_{cl}^R - P_1^R + C_1^R = 0,\\
&C_1^R = Q_1(x) + 2\alpha_1 Q_2(x) + 3\alpha_1^2 Q_3(x) + 2\alpha_1 A^T(x)P_0^R A(x)\\
&-2\alpha_1 A^T(x)P_0^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x);\\
&A_{cl}^R = \alpha_1(A(x) - B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x))
\end{aligned}$$

3)
$$\begin{aligned}
&(A_{cl}^R)^T P_2^R A_{cl}^R - P_2^R + C_2^R = 0,\\
&C_2^R = Q_2(x) + 3\alpha_1 Q_3(x) + A^T(x)P_0^R A(x) + 2A^T(x)P_1^R A(x)\alpha\\
&-A^T(x)P_0^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x)\\
&-A^T(x)P_1^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_1^R A(x)\alpha_1^2\\
&-2\alpha_1 A^T(x)P_1^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x)\\
&-2\alpha_1 A^T(x)P_0^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_1^R A(x)\\
&+A^T(x)P_1^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_1^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x)\alpha_1^2\\
&+A^T(x)P_0^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_1^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_1^R A(x)\alpha_1^2\\
&+2A^T(x)P_0^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_1^R B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x)\alpha_1
\end{aligned}$$

I. The pares of matrices ($\boldsymbol{\alpha_2} A(x),B(x)$) and ($\boldsymbol{\alpha_1} A(x),B(x)$) are controllable, and ($\boldsymbol{\alpha_2} A(x)$, $Q_0 + \boldsymbol{\alpha_2} Q_1(x) + \boldsymbol{\alpha_2}^2 Q2(x)$) , ($\boldsymbol{\alpha_1} A(x)$, $Q_0 + \boldsymbol{\alpha_1} Q_1(x) + \boldsymbol{\alpha_1}^2 Q2(x)$) are observable.

II. $A(x),\ B(x),\ Q_1(x), Q_2(x)$ *are continuous and bounded functions on X.*

III. $\left|\lambda(\alpha_1(A(x) - B(x)(R_0 + B^T(x)P_0^R B(x))^{-1} B^T(x)P_0^R A(x)))\right| < 1$

$$\left|\lambda(\alpha_2(A(x) - B(x)(R_0 + B^T(x)P_0^L B(x))^{-1} B^T(x)P_0^L A(x)))\right| < 1$$

IV. There exist such matrices $Q_j(x) > 0,$ *such that*

$C_j^{\ L}(x), C_j^{\ R}(x),\ j = 1,2$ *are positive definite matrices*

$\forall x \in X,\ \varepsilon \in [\alpha_1, \alpha_2].$

Thus matrices Q_0, Q_1, Q_2 are selected in such a way that conditions *I-III* are satisfied providing the positive definiteness of $P_2^{\ L}, P_2^{\ R}$.

THE CONSTRUCTION OF TWO-POINT PADÉ APPROXIMATION OF THE RICCATI EQUATION SOLUTION

Here, using two second-order asymptotic approximations for the solution of the Riccati equation we will construct a two-point Padé approximation of order [2/2], namely

$$PA^{[2/2]}(x,\varepsilon)=\left(\mathrm{M}_0(x)+\varepsilon\mathrm{M}_1(x)+\varepsilon^2\mathrm{M}_2(x)\right)\times\left(E+\varepsilon\mathrm{N}_1(x)+\varepsilon^2\mathrm{N}_2(x)\right)^{-1}$$

where E is an identity matrix, matrices M, N are square continuously differentiable matrices of dimension $n\times n$.

The unknown matrix coefficients are found from the system of equations that is obtained by equating the coefficients with the same powers of the parameter ε of the Padé [2/2] representation with the asymptotic expansions in the right and left neighborhoods of points $\alpha_1,\alpha_2=0$

$$\left(\mathrm{M}_0(x)+\varepsilon\mathrm{M}_1(x)+\varepsilon^2\mathrm{M}_2(x)\right)\left(E+\varepsilon\mathrm{N}_1(x)+\varepsilon^2\mathrm{N}_2(x)\right)^{-1}=P_0^{R}(x)+(\varepsilon-\alpha_1)P_1^{R}(x)+(\varepsilon-\alpha_1)^2P_2^{R}(x).$$

$$\left(\mathrm{M}_0(x)+\varepsilon\mathrm{M}_1(x)+\varepsilon^2\mathrm{M}_2(x)\right)\left(E+\varepsilon\mathrm{N}_1(x)+\varepsilon^2\mathrm{N}_2(x)\right)^{-1}=P_0^{L}(x)+(\alpha_2-\varepsilon)P_1^{L}(x)+(\alpha_2-\varepsilon)^2P_2^{L}(x),$$

The system of equations for the Padé coefficients

$$\begin{pmatrix} \mathrm{M}_0 \\ \mathrm{M}_1 \\ \mathrm{M}_2 \\ \mathrm{N}_1 \\ \mathrm{N}_2 \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & 0 & 0 \\ 0 & E & 0 & -(P_0^L + aP_1^L + a^2 P_2^L) & 0 \\ 0 & E & 0 & -(P_0^R - aP_1^R + a^2 P_2^R) & 0 \\ 0 & 0 & E & -(-P_1^L - 2aP_2^L) & -(P_0^L + aP_1^L + a^2 P_2^L) \\ 0 & 0 & E & -(P_1^R - 2aP_2^R) & -(P_0^R - aP_1^R + a^2 P_2^R) \end{pmatrix}^{-1} \begin{pmatrix} P_0^R - aP_1^R + a^2 P_2^R \\ (-P_1^L - 2aP_2^L) \\ (P_1^R - 2aP_2^R) \\ P_2^L \\ P_2^R \end{pmatrix}.$$

If matrices $Q_0(x), Q_1(x), Q_2(x)$ are chosen in such a way that matrix in the "denominator" $\left(E+\varepsilon \mathrm{N}_1(x)+\varepsilon^2 \mathrm{N}_2(x)\right)^{-1}$ is invertible for all admissible *x,ε*, then the $PA_{[2/2]}$ can be constructed and the corresponding Padé regulator is obtained.

$$u(x,\varepsilon) = -\varepsilon\left[R_0 + B(x(t))^T K^{[2/2]}(x,\varepsilon)B(x(t))\right]^{-1} \times \\ \times B(x(t))^T K^{[2/2]}(x,\varepsilon)A(x(t))x(t), \qquad (6)$$

where $K^{[2/2]}(x,\varepsilon) = \dfrac{\left(PA^{[2/2]}(x,\varepsilon)\right)^T + PA^{[2/2]}(x,\varepsilon)}{2}$

II. The $K^{[2/2]}(x,\varepsilon) > 0$ *matrix is symmetric and positive definite for* $\forall x \in X,\ \varepsilon \in [\alpha_1, \alpha_2]$.

$PA^{[2/2]}(x,\varepsilon) -$ is the constructed Padé approximation.

Numerical experiments

$$x(t+1) = \varepsilon A(x)x(t) + B(x)u(t)$$

System matrices are

$$A(x) = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 + x_1 \end{bmatrix},\ B(x) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}.$$

Criterion matrices are

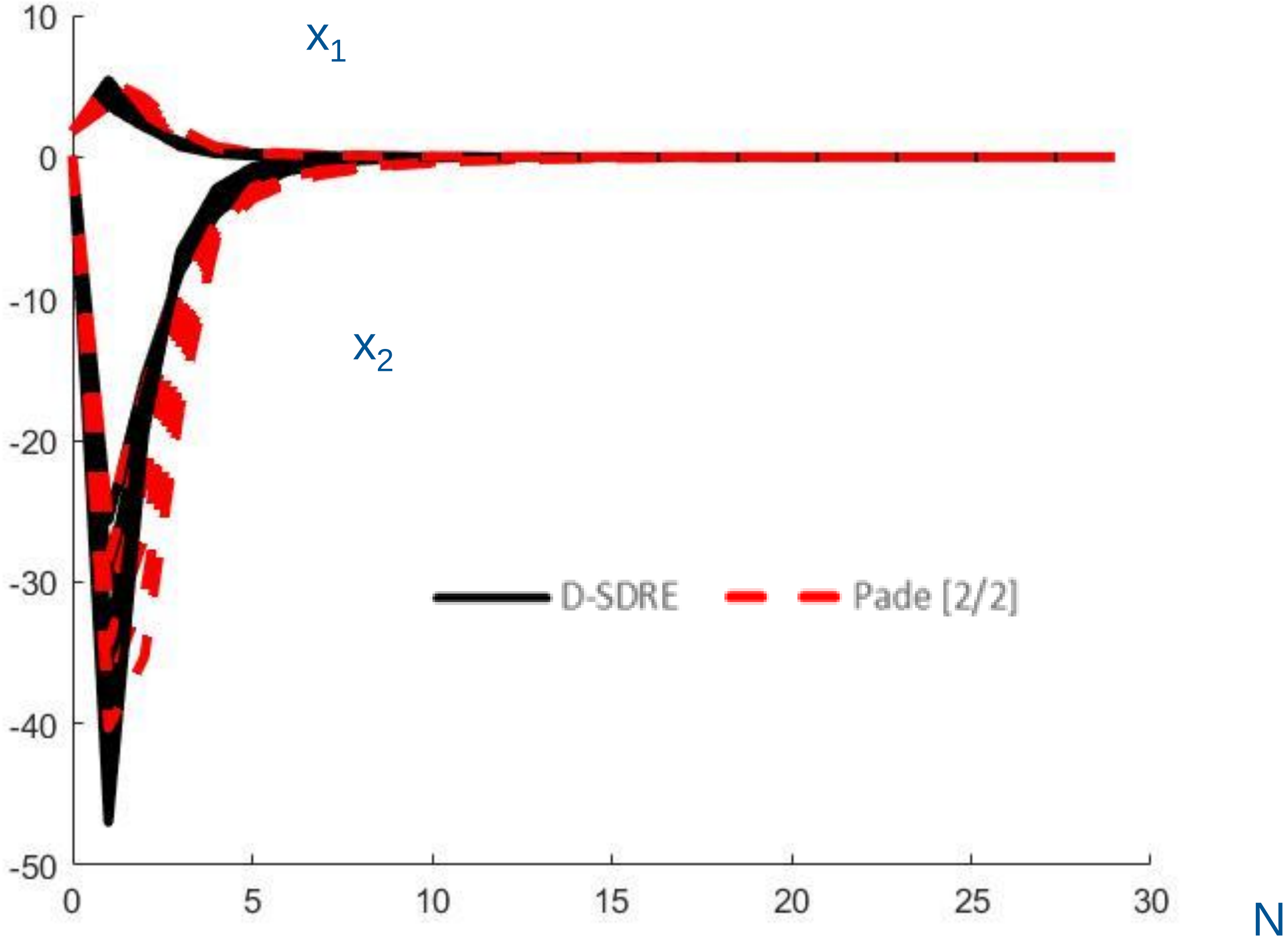
$$Q_0 = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix},\ Q_1(x) = Q_2(x) = \begin{bmatrix} 11 + 0.01{x_1}^2 & 0 \\ 0 & 11 + 0.01{x_2}^2 \end{bmatrix},\ R_0 = 1$$

Here $x(0) = (1.8 \ \ 0.1)^T$, $\alpha_1 = 2$, $\alpha_2 = 3$

$N = 30 -$ the number of steps.

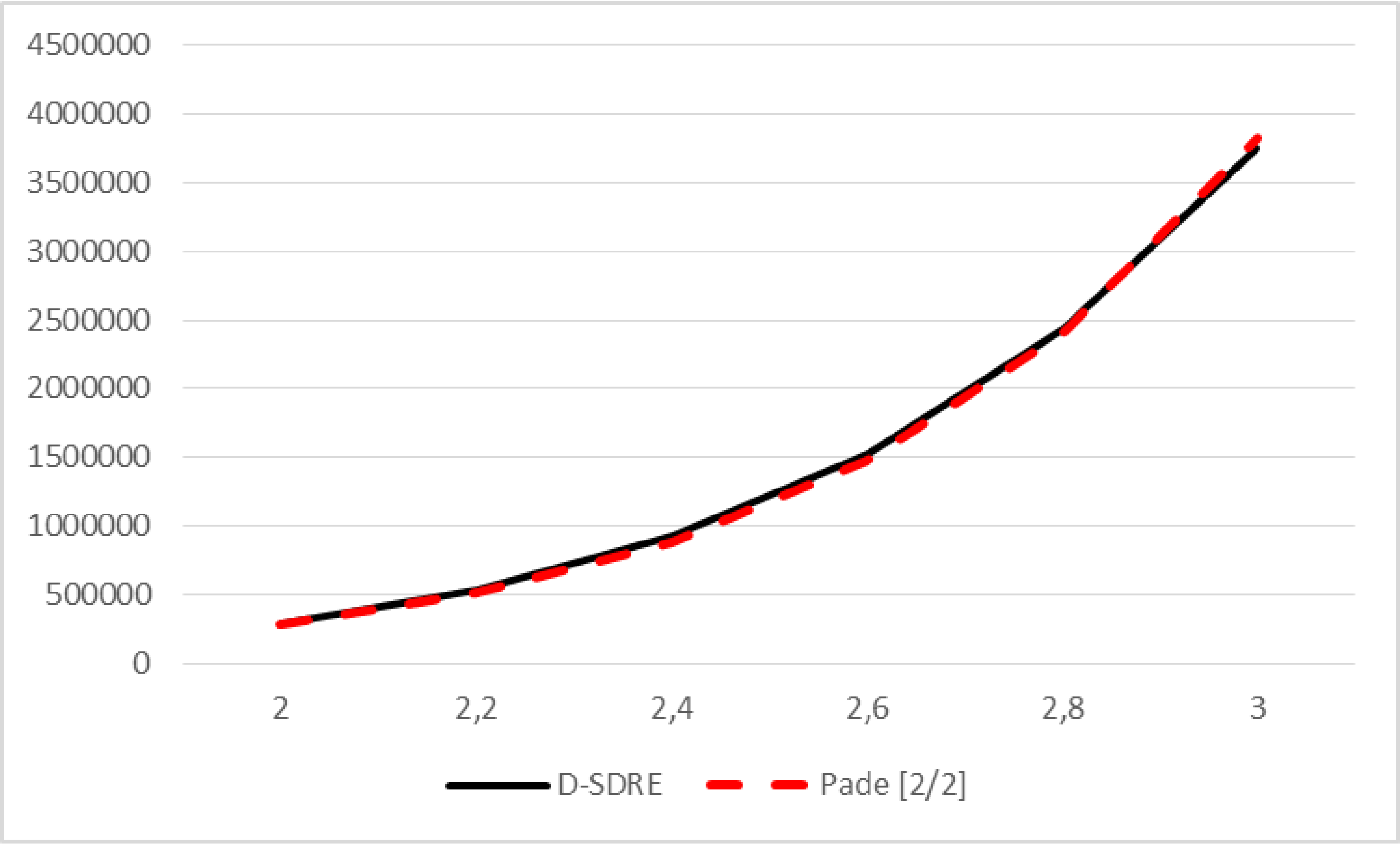
Here the comparison is made for the D-SDRE regulator and the proposed Padé regulator of order [2/2]

Closed-loop system trajectories for different ε



Criterion values along the regulators for different parameter values.

I(u)



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Conclusion

- A set of asymptotic expansions with respect to the introduced small parameter can not only serve as a basis for approximating various functions on parameter variation intervals based on the PA, but also serve as a kind of conditional basis for constructing efficient numerical algorithms.
- Asymptotic approximations of a certain order and the corrresponding PA, reflect a qualitative picture of the exact solution in some areas of parameter variation and can often serve as initial approximations in multi-extremal nonlinear programming problems that appear during the solution of complex nonlinear problems.
- The use of asymptotic approximations for feedback controls construction helps to reduce the number of calculations by several orders of magnitude while achieving the same accuracy. This is also due to the fact that the PA structure suggests directions for its modification with the help of additional optimization procedures to improve its quality.

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Thank you for your attention!