On Incomplete Rank Matrices

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Matrices are used in all areas of mathematics. The rank serves as an essential characteristic of a matrix. If K is a field and $m \times n$-matrix A over K (i.e. $A \in K^{m \times n}$), $r = \mathrm{Rank}\, A$ then the situation of incomplete rank is possible, i.e. the situation in which $d = \min\{m, n\} - r > 0$. This is an obstacle to carrying out some transformations of the matrix A and performing calculations related to A.

Rank regulation

Proposition 1

Let A be a matrix of size $m \times n$ over a field K with $r = \mathrm{Rank}\, A$ and $d = \min\{m, n\} - r > 0$, which implies that the rank of A is not full.

Then it is possible to choose algorithmically d elements from A such that, upon replacement of their values with any other values from K, yield a matrix $\tilde{A}$ of full rank (when $m = n$, the matrix $\tilde{A}$ is nonsingular).

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A matrix element, whose replacement increases the rank of the matrix, will be called a *rank-regulating element*. We prove that an incomplete rank matrix necessarily contains rank-regulating elements.

If A is of size $m \times n$, $r = \mathrm{Rank}\, A$ and $d = \min\{m, n\} - r > 0$ then A contains d rank-regulating elements belonging to different rows and columns. Those d elements have the property formulated in Proposition 1.

Example 1

Consider the following 5×5-matrix over the field of rational numbers:

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5 & -12 \\ 3 & -7 & 8 & 9 & 13 \end{bmatrix}. \tag{1}$$

Its rank is 3. It can be shown that a_{22} and a_{44} are rank-regulating elements of A.

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$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & \boxed{1} & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & \boxed{-5} & -12 \\ 3 & -7 & 8 & 9 & 13 \end{bmatrix}. \tag{2}$$

Replacing them by zeros, we obtain a matrix $\tilde{A}$ with $\det \tilde{A} = 55$;

if instead we add 1 to each of the initial a_{22}, a_{44}, then for the resulting matrix $\tilde{\tilde{A}}$ we have $\det \tilde{\tilde{A}} = -11$ (obviously, $\mathrm{Rank}\, \tilde{A} = \mathrm{Rank}\, \tilde{\tilde{A}} = 5$).

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Example 2

Consider a non-square matrix. Let A be the following 6×4-matrix over the field of rational numbers:

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 4 & 5 & 6 & 3 \\ 1 & -2 & 0 & 0 \\ 1 & 1 & 4 & 1 \end{bmatrix}. \tag{3}$$

It can be shown that any of

$$a_{44}, \quad a_{54}, \quad a_{64}$$

is a rank-regulating element of A. For example, replacing a_{54} (which is equal to 0) by 1 we obtain $\tilde{A}$ of full rank: Rank $\tilde{A} = 4 = \min\{6, 4\}$.

The following example shows that not every element is rank-regulating.

Example 3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

It can be seen that, for example, the element a_{33} does not affect the rank, unlike, say, a_{13}:

not

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & \boxed{1} \end{bmatrix},$$

but

$$\begin{bmatrix} 1 & 2 & \boxed{3} \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

Matrices over truncated formal series

Below, we consider the field K as the formal Laurent series field $F((x))$ over a field F. (The field $F((x))$ is the quotient field of the formal power series ring $F[[x]]$.)

Let the elements of the matrix A be polynomials, which are considered as truncated power series. If $\det A = 0$ then A has obviously a prolongation which is a singular matrix belonging to $F[[x]]^{n\times n}$: such a prolongation can be obtained by adding to each element of A an infinite sequence of zero terms.

On the other hand, using the recipe from Proposition 1, we can construct a prolongation which gives a nonsingular matrix $\tilde{A}$. To do this, we can, for example, add to each of the rank-regulating elements some terms that have degrees higher (say, by 1) than the degrees of the elements of the matrix A.

Some important concepts

The *degree* of a polynomial $p(x) = \sum p_i x^i$ (of an element of $K[x]$ or $K[x^{-1}, x]$) is defined by $\deg p(x) = \max\{i \mid p_i \neq 0\}, \deg 0 = -\infty$. The degree of a polynomial matrix A is equal to the largest of the degrees of the elements of this matrix.

Given a series $a(x)$ and a *truncation degree* $l \in \mathbb{Z} \cup \{-\infty\}$, the *$l$-truncation* of $a(x)$ is obtained from $a(x)$ by discarding all its terms of degree exceeding l. For a matrix $M \in K[[x]]^{n \times n}$ and a given truncation degree l, the l-truncation of M is obtained by replacing all the entries of M by their l-truncations.

A *prolongation* of the polynomial $p(x)$ with $\deg p(x) = d$ is any series of the form $p(x) + x^{d+1} q(x)$, $q(x) \in K[[x]]$. A prolongation of a matrix $A \in K[x]^{n \times n}$ with $\deg A = d$ is any matrix M of the form $A + x^{d+1} B$, where $B \in K[[x]]^{n \times n}$ (if $d = -\infty$, i.e. $A = 0$ then the factor x^{d+1} should be ignored.)

We consider a system $Ay = 0$ in which the nonzero polynomial matrix A of degree l is the l-truncation of an unknown matrix $M \in K[[x]]^{n\times n}$. Can we expect that the system $My = 0$ has non-zero solutions in $K((x))^n$? We show that by examining a given matrix A, one can algorithmically find out which of the following two cases takes place:

C_1 Under no prolongation of the matrix A does the system $My = 0$ have nonzero solutions in $K((x))^n$.

C_2 Depending on the particular prolongation of the matrix A, the system $My = 0$ may or may not have a nonzero solution in $K((x))^n$.

It is noteworthy that these two cases do not include the possibility where the system $My = 0$ has a non-zero solution for all prolongations of the matrix A, which would happen if the matrix A
were singular for all possible prolongations. The fact is that for a matrix $A = [a_{ij}] \in F[x]^{n\times n}$, $\det A = 0$, there exists its non-singular polynomial prolongation

$$\tilde{A} = [\tilde{a}_{ij}] \in F[x]^{n\times n}, \ \det \tilde{A} \neq 0.$$

On the initial segments of series solutions invariant under the prolongation of the matrix A

It is easy to see that the following corollary of Proposition 1 holds:

Corollary 1

Let $A \in K[[x]]^{n \times n}$ with $\det A = 0$. Then there exists no nonzero $p \in K[x^{-1}, x]^n$ such that for any matrix M that is a prolongation of the matrix A, the system $My = 0$ has a solution in the form of a prolongation of p belonging to $K((x))^n$.

Indeed, A has a prolongation M such that the system $My = 0$ has no nonzero solutions.

It can be noted here that this situation differs from that observed in connection with linear homogeneous differential equations with truncated power series in the role of coefficients. It can be shown that for the operator

$$L = (-1 + x + x^2)\,\theta^2 + (-2 + x^3)\,\theta + (x + 6x^2), \;\; \theta = x\frac{d}{dx}, \qquad (4)$$

the equation $\tilde{L}y = 0$, for any prolongation $\tilde{L}$ of the coefficients of L, has a Laurent-series solution with valuation -2, namely

$$x^{-2} - 5\,x^{-1} + 1 + O(x)\ .$$

Thus, here a desired Laurent polynomial is, e.g.,

$$p = x^{-2} - 5\,x^{-1} + 1.$$

Similar examples were considered in

S. Abramov, D. Khmelnov, and A. Ryabenko. 2019. Linear Ordinary Differential Equations and Truncated Series. *Computational Mathematics and Mathematical Physics* 59 (2019), 1649–1659.

It is pertinent here to note that in the differential case, the operator $L \in K[[x]][\theta]$ and the differential equation $Ly = 0$ are associated with an algebraic equation called the *indicial equation* — see, e.g., Ch. 9, §8 in the book

E. A. Coddington and N. Levinson. 1955. *Theory of Ordinary Differential Equations.* McGraw, Hill, New York.

If $Ly = 0$ has a Laurent-series solution with valuation ν, then ν is an integer root of the corresponding indicial equation.

The indicial equation does not depend on the terms of high powers in x appearing in the coefficients of the operator L.

It turns out that it is possible to establish whether the equation $Ly = 0$ has a Laurent-series solution with a chosen valuation ν, and in the case when such a solution exists, to construct the first few terms of such a solution, using for this purpose only the terms of some low degrees entering into the coefficients of the operator L.

Thus, the terms of a sufficiently high degree in the coefficients of the operator do not play any role in testing the existence of Laurent-series solutions and do not affect the initial terms of Laurent-series solutions. But note that changing the coefficients of the operator by adding terms of lesser degree can significantly change the picture.

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E.g., by changing the operator

$$L = (-1 + x + x^2)\,\theta^2 + (-2 + x^3)\,\theta + (x + 6x^2)$$

to

$$\bar{L} = (-1 + x + x^2)\,\theta^2 + (-2 + x^3)\,\theta + (1 + x + 6x^2),$$

we get the equation $\bar{L}y = 0$, which has no Laurent-series solutions. However, L is equal to (4), and it has been noted that (4) has Laurent-series solutions.

In the previous sections of our presentation, the "sensitivity" of systems of linear algebraic equations to changes in the coefficients of terms of high degrees was shown. At the same time, as we see, in the differential case, even for scalar linear equations, "sensitivity" to changes in terms of small powers is manifested.