Bounding the Support in the Differential Elimination Problem

Mukhina Yulia

Joint work with Gleb Pogudin

Department of Higher Algebra, Lomonosov Moscow State University

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Differential ring

Definition

A differential ring $(R, ')$ is a commutative ring with a derivation $' : R \to R$, that is, a map such that,
for all $a, b \in R$, $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$.
A differential field is a differential ring that is a field.
For $i > 0$, $a^{(i)}$ denotes the i-th order derivative of $a \in R$.

Let x be an element of a differential ring. We introduce
$x^{(\infty)} := (x, x', x'', x^{(3)}, \ldots)$.

Ring of differential polynomials

Definition

Let R be a differential ring. Consider a ring of polynomials in infinitely many variables

$$R[x^{(\infty)}] := R[x, x', x'', x^{(3)}, \ldots]$$

and extend the derivation from R to this ring by $(x^{(j)})' := x^{(j+1)}$. The resulting differential ring is called *the ring of differential polynomials in x over R*.

The ring of differential polynomials in several variables is defined by iterating this construction.

Differential ideal

Definition

Let $S := R[x_1^{(\infty)}, \dots, x_n^{(\infty)}]$ be a ring of differential polynomials over a differential ring R. An ideal $I \subset S$ is called *a differential ideal* if $a' \in I$ for every $a \in I$.

One can verify that, for every $f_1, \dots, f_s \in S$

$$\langle f_1^{(\infty)}, \dots, f_s^{(\infty)} \rangle$$

is a differential ideal. Moreover, this is the minimal differential ideal containing $f_1, \dots, f_s$, and we will denote it by $\langle f_1, \dots, f_s \rangle^{(\infty)}$.

Differential ideal

Definition

Let I be an ideal in ring R, and $a \in R$. Then

$$I : a^\infty := \{b \in R \mid \exists N : a^N b \in I\}.$$

Note that the resulting set $I : a^\infty$ is also an ideal in R. And if I is a differential ideal, than set $I : a^\infty$ is also a differential ideal.

The order of differential polynomial

Definition

For every $1 \leq i \leq n$, we will call the largest j such that $x_i^{(j)}$ appears in P *the order of P respect to x_i* and denote it by $\operatorname{ord}_{x_i} P$; if P does not involve x_i, we set $\operatorname{ord}_{x_i} P := -1$.

The order of differential polynomial

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Example

For differential polynomial

$$P = (x')^2 - 4x^3 + x$$

we have

$$\operatorname{ord}_x P = 1, \; \frac{\partial P}{\partial x'} = 2x'.$$

Problem

Consider a system of differential equations of the form

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is a tuple of differential indeterminates and $\mathbf{f} = (f_1, ..., f_n)$ is a tuple of polynomials from $\mathbb{C}[\mathbf{x}]$.
One natural elimination task is to eliminate all the variables in the system (1) except one, say x_1, that is, describe a differential ideal

$$\langle x_1' - f_1(\boldsymbol{x}), ..., x_n' - f_n(\boldsymbol{x})\rangle^{(\infty)} \cap \mathbb{C}[x_1^{(\infty)}]. \tag{2}$$

Problem

The ideal (2) is uniquely determined by its minimal polynomial $f_{\min}$ (polynomials are compared first w.r.t. the order and then w.r.t. total degree)

$$i_1 > i_2 \Rightarrow x_1^{(i_1)} > x_1^{(i_2)}$$

We can define ideal (2) as

$$I = \langle f_{\min} \rangle^{(\infty)} : H^{(\infty)},$$

with

$$H = \frac{\partial f_{\min}}{\partial x_1^{(h)}} \text{ and } h = \operatorname{ord}_{x_1} f_{\min}.$$

Toy example

Consider the following model:

$$\begin{cases} x_1' = x_2^2, \\ x_2' = x_2. \end{cases}$$

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In this case,

$$I = \langle x_1' - x_2^2, x_2' - x_2 \rangle^{(\infty)} \cap \mathbb{C}[x_1^{(\infty)}].$$

is uniquely determined by

$$f = 2x_1' - x_1''$$

How it works

Consider the following model:

$$\begin{cases} x_1' = x_1^2 + x_2^2, \\ x_2' = x_2 + 1. \end{cases}$$

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is uniquely determined by

$$f = (x_1'')^2 + 4x_1^4 + 4x_1^2(x_1')^2 + 4x_1^2 + 4(x_1')^2 - 8x_1^2x_1' - 4x_1x_1'x_1'' - 4(x_1' - x_1^2)x_1'' + \\ + 8x_1x_1'(x_1' - x_1^2) - 4x_1'.$$

Motivation

Describe/compute the minimal polynomial of the elimination ideal $\rightarrow$

$\rightarrow$ find the support of the minimal polynomial

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Describe/compute the minimal polynomial of the elimination ideal $\rightarrow$

$\rightarrow$ find the support of the minimal polynomial

$\triangleright$ finding truncated power series solutions of $\mathbf{x}' = \mathbf{g}(\mathbf{x})$

Problem

Consider the case of system

$$\begin{cases} x_1' = g_1(x_1, x_2), \\ x_2' = g_2(x_1, x_2), \end{cases}$$

where $g_1, g_2 =$ generic polynomials of degrees d_1 and d_2.

Small step for a man

Theorem

Consider the system

$$\begin{cases} x_1' = g_1(x_1, x_2), \\ x_2' = g_2(x_1, x_2), \end{cases}$$

where $g_1, g_2 =$ *generic polynomials of degrees* 2 *and* d. *Then the Newton polytope of the minimal polynomial in* (s_0, s_1, s_2)*-coordinates* $(x_1^{s_0}(x_1')^{s_1}(x_1'')^{s_2})$ *is*

1. *a pyramid with vertices* $(0,0,0), (4,0,0), (2,2,0), (0,3,0), (0,0,2)$ *if* $d = 1$,
2. *a tetrahedron with vertices* $(0,0,0), (2(d+1),0,0), (0,d+1,0), (0,0,2)$ *if* $d \geq 2$.

Newton polytope of the minimal polynomial

Figure: Newton polytope of the minimal polynomial for $(2, d)$ case.

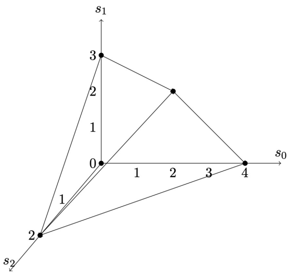

Figure: $(2, 1)$ case

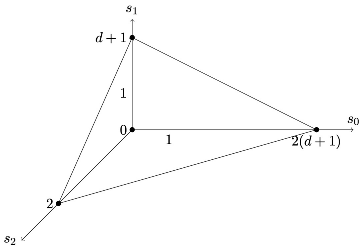

Figure: $(2, d)$, $d \geq 2$ case

General case

Theorem

Consider the system

$$\begin{cases} x_1' = g_1(x_1, x_2), \\ x_2' = g_2(x_1, x_2), \end{cases}$$

where $g_1, g_2 =$ *generic polynomials of degrees* d_1 *and* d_2*. Then the Newton polytope of the minimal polynomial in* (s_0, s_1, s_2)*-coordinates* $(x_1^{s_0}(x_1')^{s_1}(x_1'')^{s_2})$ *is*

1. *a pyramid with vertices* $(0,0,0), (d_1(d_1+d_2-1), 0, 0), (d_1(d_1-1), d_1, 0), (0, 2d_1-1, 0), (0,0,d_1)$ *if* $d_1 > d_2$*,*
2. *a tetrahedron with vertices* $(0,0,0), (d_1(d_1+d_2-1), 0, 0), (0, d_1+d_2-1, 0), (0,0,d_1)$ *if* $d_1 \leq d_2$*.*

Figure: Newton polytope of the minimal polynomial for (d_1, d_2) case.

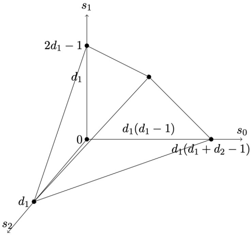

Figure:
$d_1 > d_2$ case

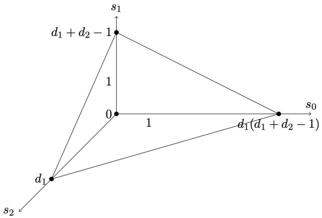

Figure:
$d_1 \leq d_2$ case

Proof

Proof plan

1. $\operatorname{ord} f_{\min} = 2.$
2. $f_3 = (f_1)' = 0 \to$

$$\begin{cases} f_1 = x_1' - g_1(x_1, x_2), \\ f_3 = x_1'' - x_1' \dfrac{\partial}{\partial x_1} g_1(x_1, x_2) - g_2(x_1, x_2) \dfrac{\partial}{\partial x_2} g_1(x_1, x_2). \end{cases} \quad (3)$$

3. $\operatorname{Res}_{x_2}(f_1, f_3) = (f_{\min})^n, \ n \in \mathbb{N}$
4. $\operatorname{Res}_{x_2}(f_1, f_3)$ is irreducible + $\deg \operatorname{Res}_{x_2}(f_1, f_3) = d \Rightarrow$
$\Rightarrow$ for $(x_1)^{s_0}(x_1')^{s_1}(x_1'')^{s_2}$ in $\operatorname{Res}_{x_2}(f_1, f_3)$

$$s_0 + s_1 + s_2 \leq d.$$

$(1, d)$ case

Consider the system

$$\begin{cases} x_1' &= g_1(x_1, x_2) = c_1x_1 + c_2x_2 + c_3, \;\; c_2 \neq 0, \\ x_2' &= g_2(x_1, x_2), \end{cases}$$

where $g_1, g_2 =$ polynomials of degrees 1 and d. In this case

$$x_1'' = c_1x_1' + c_2x_2'.$$

Consider new system

$$\begin{cases} f_1 &= x_1' - c_1x_1 - c_2x_2 - c_3, \\ f_3 &= x_1'' - c_1x_1' - c_2g_2(x_1, x_2). \end{cases}$$

Compute the resultant of polynomials f_1 and f_3 to eliminate x_2.

$$\mathrm{Res}_{x_2}(f_1, f_3) = x_1'' - c_1x_1' - c_2g_2(x_1, \frac{1}{c_2}(x_1' - c_1x_1 - c_3)).$$

$(1, d)$ case

In this case

1. $\mathrm{Res}_{x_2}(f_1, f_3) = f_{\min}$.
2. $\deg \mathrm{Res}_{x_2}(f_1, f_3) = d \Rightarrow$ for $(x_1)^{s_0}(x_1')^{s_1}(x_1'')^{s_2}$ in $\mathrm{Res}_{x_2}(f_1, f_3)$ we have

$$s_0 + s_1 + s_2 \leq d.$$

3. $\mathrm{Res}_{x_2}(f_1, f_3)$ contains $x_1'', (x_1')^d$ и $(x_1)^d$.

$(1, d)$ case

Then the Newton polytope of the minimal polynomial in (s_0, s_1, s_2)-coordinates $(x_1^{s_0}(x_1')^{s_1}(x_1'')^{s_2})$ is a tetrahedron with vertices

$$(0,0,0),(d,0,0),(0,d,0),(0,0,1)$$

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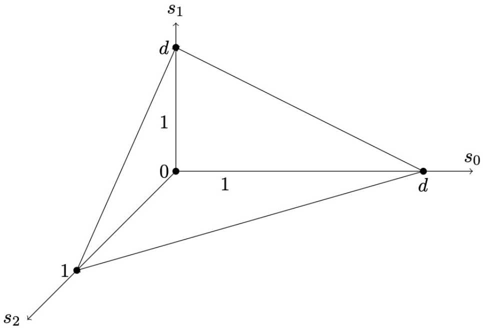

Figure: Newton polytope of the minimal polynomial for $(1, d)$ case.

Thank you for your attention!