Computing of tropical sequences associated with Somos sequences in Gfan package

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5th International Conference "Computer Algebra", Moscow, June 27, 2023

Goal of the work

The main goals of this work are

- computing finite tropical sequences associated with Somos sequences using the Gfan package;
- testing D.Yu. Grigoriev hypothesis.

For a set of tropical recurrent sequences, D.Yu. Grigoriev put forward a ***hypothesis*** of stabilization of the maximum dimensions of solutions to systems of tropical equations given by polynomials, which depend on the length of the sequence under consideration.

Gfan is a software package for computing universal Gröbner bases, some related geometric objects (Gröbner fans) and tropical varieties, developed in 2005 by A. Jensen, based on the algorithms described and developed in his dissertation.

Tropical semiring

Tropical semiring is a semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, with tropical operations

$$x \oplus y := \max(x, y), \quad x \otimes y := x + y$$

One of the main differences between tropical and classical mathematics is that tropical addition is ***idempotent***

$$x \oplus x = \max(x, x) = x$$

It is important to note that there is no tropical subtraction operation in a tropical semiring. The equation $a \oplus x = -\infty$ has no solution for any a, except $-\infty$.

Tropical polynomials

Let $x_1, \ldots, x_n$ be variables that are elements of the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. By commutativity, we can sort the multiplication and write the ***tropical monomial*** in classical notation:

$$q(x_1, \ldots, x_n) = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n} = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n.$$

A ***tropical polynomial*** is a finite linear combination of tropical monomials

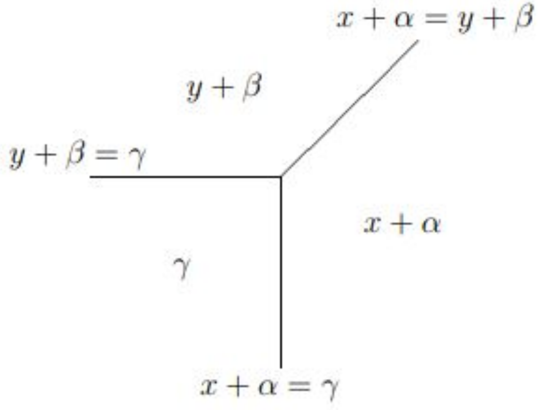
$$p(x_1, \ldots, x_n) = \bigoplus_j \left(a_j \otimes x_1^{\otimes i_{j1}} \otimes \cdots \otimes x_n^{\otimes i_{jn}} \right) = \max_j \left(a_j + i_{j1} x_1 + \cdots + i_{jn} x_n \right).$$

$x_1, \ldots, x_n$ are ***tropical zeros (roots)*** of the polynomial *p*, if the maximum of the tropical monomials q_j is reached at least in two different values of *j*.

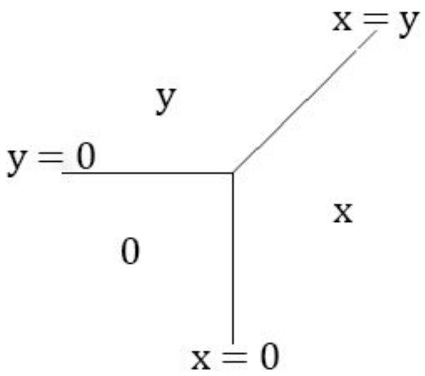
Tropical line

Let's consider a tropical line as an example:

$$f(x,y) = \alpha \otimes x \oplus \beta \otimes y \oplus \gamma = \max(\alpha + x, \beta + y, \gamma).$$



Tropical line
$f(x,y) = \alpha \otimes x \oplus \beta \otimes y \oplus \gamma$



Tropical line
$f(x,y) = x \oplus y \oplus 0$

Tropical hypersurface

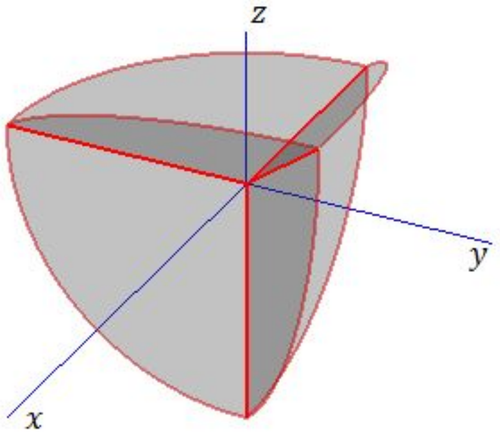
Tropical hypersurface $\mathcal{T}(f)$ of polynomial f is the set of tropical zeros of the tropical polynomial $Trop(f)$. Tropicalization of a polynomial $Trop(f)$ is a conversation from classical operations to their tropical analogues, the coefficients at monomials are assumed to be equal to zero.

As an example, consider the tropical hypersurface of the polynomial *g*

$$g = x + 2y + z + 1$$

$$Trop(g) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 0$$

$$Trop(g) = \max(x, y, z, 0)$$



Tropical hypersurface $\mathcal{T}(g)$

Tropical prevariety

A ***tropical prevariety*** of a polynomial system $f_1, \ldots, f_n$ is a finite intersection of tropical hypersurfaces $\mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_n)$.

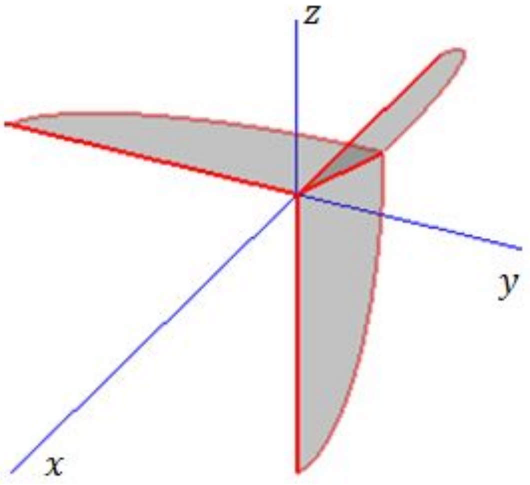
For example, consider tropical hypersurfaces *g* and *h*

$$g = x + 2y + z + 1$$

$$Trop(g) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 0$$

$$h = x + y + 2z$$

$$Trop(h) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z$$



Tropical prevarity g, h

Classical Somos sequences

Let $k \geq 2$ be natural and

$$\alpha = \{\alpha_i | 1 \leq i \leq [k/2]\}, \quad x = \{x_j | -k/2 < j \leq [k/2]\}$$

are two sets of independent formal variables in the amount of [k/2] in the first case and k in the second. ***The sequence of rational Somos-k functions*** of variables from α and x, $S_k(n) = S_k(n;\alpha;x)(n \in \mathbb{Z})$ is determined by the recursive relation

$$S_k\left(n + \left[\frac{k+1}{2}\right]\right) S_k\left(n - \left[\frac{k}{2}\right]\right) = \sum_{1 \leq i \leq k/2} \alpha_i S_k\left(n + \left[\frac{k+1}{2}\right] - i\right) S_k\left(n - \left[\frac{k}{2}\right] + i\right).$$

For example, for $k = 4, \alpha_1 = \alpha_2 = 1$, the recurrence relation has the form

$$S_4(n+2)S_4(n-2) = S_4(n+1)S_4(n-1) + 2S_4(n)$$

Tropical Somos sequences

Tropical sequence Somos-k satisfies the recurrence relation

$$p_k\left(n+\left[\frac{k+1}{2}\right]\right)+p_k\left(n-\left[\frac{k}{2}\right]\right)=\min_{1\le i\le k/2}\left\{p_k\left(n+\left[\frac{k+1}{2}\right]-i\right)+p_k\left(n-\left[\frac{k}{2}\right]+i\right)\right\}.$$

An interesting fact is that the tropical analogue of such sequences is related to the classical Somos sequences by the following relation

$$S_k(n)=\left(\prod_{-k/2<j\le[k/2]} x_j^{p_k^{(j)}(n)}\right)P_k(n),$$

where $P_k(n)=P_k(n;\alpha;x)$ are polynomials with integer coefficients and $p_k^{(j)}(n)$ are integer sequences.

Tropical sequences Somos-4

For $k = 4$

$$p_4(n+2) + p_4(n-2) = \min\{p_4(n+1) + p_4(n-1), 2p_4(n)\}$$

To compute sequences change the variable

$$q_4(n) = \Delta^2 p_4(n) = \Delta p_4(n+1) - \Delta p_4(n) = p_4(n+2) - 2p_4(n+1) + p_4(n).$$

We get

$$q_4(n-1) + q_4(n) + q_4(n+1) + \max\{0, q_4(n)\} = 0$$

Let $y_n = q_4(n)$, bring the addition to the maximum

$$\max\{y_{n-1} + y_n + y_{n+1}, y_{n-1} + 2y_n + y_{n+1}\} = y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1}$$

Since the tropical prevariety is computed not by equal zero, we add zero as a tropical monomial to the polynomial and exclude the rays and cones of the original

$$y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1} \oplus 0$$

Tropical sequences Somos-5

Let $k = 5$

$$q_5(n-2) + q_5(n-1) + q_5(n) + q_4(n+1) + \max\{0, q_5(n-1) + q_5(n)\} = 0$$

Let $y_n = q_5(n)$, bring the addition to the maximum

$$y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1}$$

Next, we calculate tropical prevarieties in the package Gfan for finite Somos sequences $y = \{y_0, \ldots, y_s\}$ that satisfy for all $2 \leq n \leq s-1$.

Results

Let d_s be the dimension of solution space.

The dimensions of the solution space for $k = 4$

s	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
d_s	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6	6	6

The dimensions of the solution space for $k = 5$

s	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
d_s	3	3	3	4	4	4	4	4	5	5	6	6	6	6	6	7	7	8	8	8

Thank you for your attention