The first differential approximation on the example of the Van der Pol oscillator

Y.A. Blinkov

Saratov State University, Russia
e-mail: blinkovua@info.sgu.ru

28 June 2023

Systems of ordinary differential equations depending on parameters is considered, using the Van der Pol oscillator as an example. Advantages of the first differential approximation method and its implementation in the computer algebra systems are discussed. It is shown that, the presented method allows to estimate the stiffness of the Van der Pol oscillator and error of numerical methods and to propose simple criteria for choosing a step in calculations. The presented implementation of the method use a standard tools of computer algebra and can be applied systems with a polynomial right-had side.

In the 60s of the last century, N. N. Yanenko [1] formulated the differential approximations method to investigate difference schemes. The main idea of this method is to replace the investigation of the properties of a difference scheme by the investigation of some problem with differential equations occupying an intermediate position between the original differential problem and the difference scheme approximating it.

First Differential Approximation (FDA)

for PDEs of evolutionary type and in particular the Korteweg-de Vries equation using computer algebra systems is discussed in [2].
In [3] FDA is reviewed for difference schemes describing ordinary differential equations. The connection between the singular perturbation of the original system and the concept of FDA is discussed. For this simple case, a relationship is shown between the method for estimating the approximation error of the solution based on the FDA analysis and the Richardson-Kalitkin method.

It should be noted, that a consistent system of PDEs can be approximate by inconsistent difference systems of equations. Examples are given in [3]. As a way to check the consistency of a system of difference equations, it is proposed to check the consistency of the FDA for a difference system. The issues of FDA calculation in computer algebra systems, Sage and SymPy are considered.

This presentation considers systems of ordinary differential equations (ODE) depending on parameters, using the Van der Pol oscillator as an example.
There are many numerical methods for solving ODE. The usage of the FDA allows to get information about the quality of the selected numerical method for a particular system using only symbolic calculations. In this presentation, Runge-Kutta methods and some multi-step methods will be considered.

The use of the FDA for their study allows obtaining information about the quality of the selected numerical method for a particular system using only symbolic calculations.

Boundary value problem for 2nd order ODE

$$\frac{d^2u}{dx^2} - u = 2x, \quad u(0) = 0, \quad u(1) = -1. \tag{1}$$

Exact solution

$$u(x) = \frac{e^x - e^{-x}}{e^1 - e^{-1}} - 2x \tag{2}$$

Consider the difference scheme

$$\frac{u_{j+2} - 2u_{j+1} - u_j}{h^2} - u_{j+1} - 2x_{j+1} = 0 \tag{3}$$

Expansion at the point $t=2$

$$F:\quad u_{tt}-u-2t+h\left(-u_{ttt}+u_t+2\right)+ \\ +h^2\left(\frac{7u_{tttt}-6u_{tt}}{12}\right)+h^3\left(\frac{u_{ttt}}{6}\right)+\mathcal{O}(h^4)=0 \tag{4}$$

$$F+hF_t:\quad u_{tt}-u-2t+h^2\left(-\frac{5u_{tttt}-6u_{tt}}{12}\right)+ \\ +h^3\left(\frac{7u_{ttttt}-4u_{ttt}}{12}\right)+\mathcal{O}(h^4)=0 \tag{5}$$

$$F+hF_t+h^2F_{tt}\frac{5}{12}:\quad u_{tt}-u-2t+h^2\left(\frac{u_{tt}}{12}\right)+ \\ +h^3\left(\frac{2u_{ttttt}+u_{ttt}}{12}\right)+\mathcal{O}(h^4)=0 \tag{6}$$

$$F+hF_t+h^2\frac{5F_{tt}-F}{12}:\quad u_{tt}-u-2t+h^2\left(\frac{2t+u}{12}\right)+ \\ +h^3\left(\frac{2u_{ttttt}+2u_{ttt}-u_t-2}{12}\right)+\mathcal{O}(h^4)=0 \tag{7}$$

Expansion at the point $t = -\frac{1}{2}$

$$F: \quad u_{tt} - u - 2t + h\left(\frac{3\left(u_{ttt} - u_t - 2\right)}{2}\right) + \\ + h^2\left(\frac{29u_{tttt} - 27u_{tt}}{24}\right) + h^3\left(-\frac{9u_{ttt}}{16}\right) + \mathcal{O}(h^4) = 0 \tag{8}$$

$$F - hF_t\frac{3}{2}: \quad u_{tt} - u - 2t + h^2\left(-\frac{25u_{tttt} - 27u_{tt}}{24}\right) + \\ + h^3\left(-\frac{29u_{ttttt} - 18u_{ttt}}{16}\right) + \mathcal{O}(h^4) = 0 \tag{9}$$

$$F - hF_t\frac{3}{2} + h^2F_{tt}\frac{25}{24}: \quad u_{tt} - u - 2t + h^2\left(\frac{u_{tt}}{12}\right) + \\ + h^3\left(-\frac{4u_{ttttt} + 7u_{ttt}}{16}\right) + \mathcal{O}(h^4) = 0 \tag{10}$$

$$F - hF_t\frac{3}{2} + h^2\frac{25F_{tt} - 2F}{24}: \quad u_{tt} - u - 2t + h^2\left(\frac{2t + u}{12}\right) + \\ + h^3\left(-\frac{4u_{ttttt} + 9u_{ttt} - 2u_t - 4}{16}\right) + \mathcal{O}(h^4) = \tag{11}$$

Boundary value problem for 2nd order ODE

$$\frac{d^2u}{dx^2} - u = 2x, \quad u(0) = 0, \quad u(1) = -1. \tag{12}$$

Exact solution

$$u(x) = \frac{e^x - e^{-x}}{e^1 - e^{-1}} - 2x \tag{13}$$

Consider the difference scheme

$$\frac{u_{j+2} - 2u_{j+1} - u_j}{h^2} - u_{j+1} - 2x_{j+1} = 0 \tag{14}$$

Behavior of the error on the exact solution

$$h^2 \left(\frac{2t + u}{12} \right) = h^2 \frac{e^t - e^{-t}}{12\,(e - e^{-1})} \tag{15}$$

Let's write Van der Pol oscillator[5]

$$u_{tt} - \mu(1 - u^2)u_t + u = 0 \tag{16}$$

as the system of two equations

$$\begin{cases} u_t - u_1 = 0, \\ u_{1\,t} - \mu(1 - u^2)u_1 + u = 0. \end{cases} \tag{17}$$

Applying the fourth-order Runge-Kutta method [6] to (17) we get

$$\begin{cases} u_1 + h\left(-\frac{\mu u_1 (u-1)(u+1)}{2} - \frac{u}{2}\right) + \ldots = 0, \\ -\mu u_1 (u-1)(u+1) - u + h\left(\frac{\mu^2 u_1 (u-1)^2 (u+1)^2}{2} + \right. \\ \qquad \left. + \frac{\mu u \left(u^2 - 2u_1^2 - 1\right)}{2} - \frac{u_1}{2}\right) + \ldots = 0. \end{cases} \tag{18}$$

Taylor expansion for $t = 0$ for (18) gives

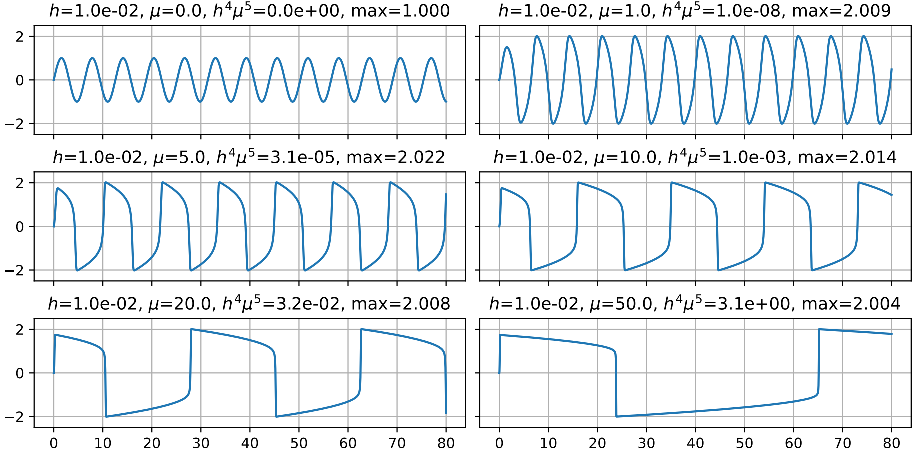
$$\begin{cases} u_t - u_1 + h\left(\frac{\mu u_1\,(u-1)\,(u+1)}{2} + \frac{u + u_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ u_{1\,t} - \mu u_1\left(1 - u^2\right) + u + h\left(-\frac{\mu^2 u_1\,(u-1)^2\,(u+1)^2}{2} - \right. \\ \qquad \left. - \frac{\mu u\left(u^2 - 2u_1^2 - 1\right)}{2} + \frac{u_1 + u1_{tt}}{2}\right) + \mathcal{O}(h^2) = 0. \end{cases} \tag{19}$$

Using algorithms for constructing Gröbner bases for series in a computer algebra system SymPy, we build FDA for (19)

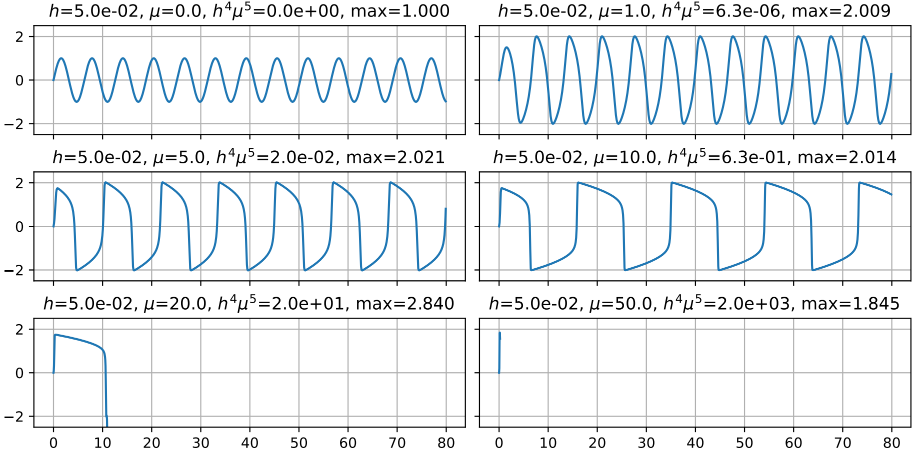
$$\begin{cases} u_t - u_1 + h^4 \left(\frac{\mu^4 u_1 (u-1)^4 (u+1)^4}{120} + \ldots \right) + \mathcal{O}(h^5) = 0, \\ u_{1\,t} - \mu u_1 \left(1 - u^2\right) + u + h^4 \left(-\frac{\mu^5 u_1 (u-1)^5 (u+1)^5}{120} - \ldots \right) + \mathcal{O}(h^5) = \end{cases} \quad (20)$$

The construction of the FDA, which is independent of the Taylor expansion point, made it possible to correctly determine the order of the numerical method and its remainder term. We built the FDA for various explicit and implicit Runge-Kutta methods, Adams–Bashforth and Adams–Moulton multi-step methods. All results are shown that the remainder terms of the second equation of the system (17) have a form $h^p(C\mu^{p+1}(u^2-1)^{p+1}) + \ldots)$, where p is the order of the method, C is some constant. The remainder term shows the stiffness of the [7] system with respect to the μ parameter. As a result, it is necessary to choose the step h in such a way as to ensure the smallness of the remainder term.

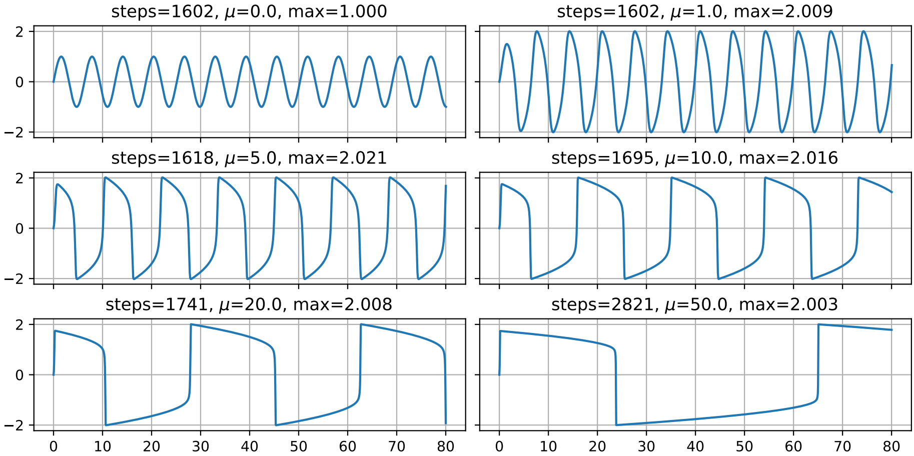
"original"Runge-Kutta $h = 0.01$



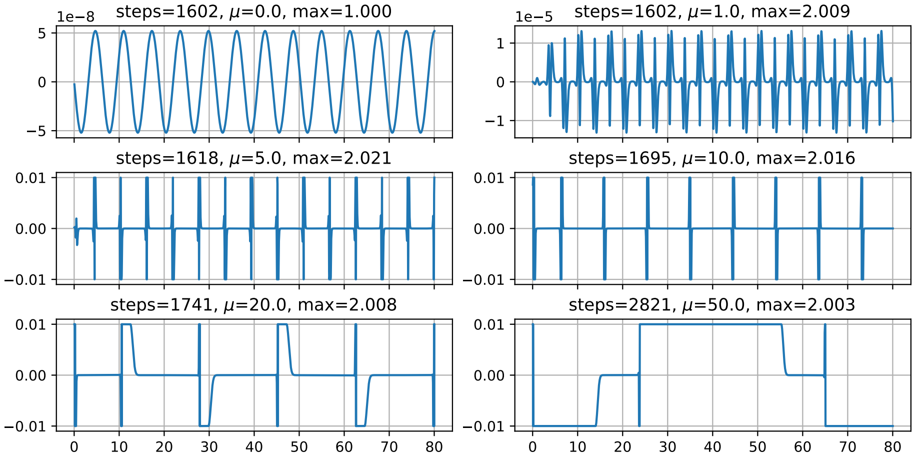
"original"Runge-Kutta $h = 0.05$



"original"Runge-Kutta $h = 0.05$



"original"Runge-Kutta $h = 0.05$



Gauss-Legendre method FDA

$$\begin{cases} -u_1 + u_t + h^2 \left(-\frac{\mu^2 u_1 (u-1)^2 (u+1)^2}{12} - \frac{\mu u \left(u^2 - 2u_1^2 - 1 \right)}{12} + \frac{u_1}{12} \right) + \mathcal{O}(h^4) = 0, \\ \mu u_1 \left(u - 1 \right) \left(u + 1 \right) + u + u_{1\,t} + h^2 \left(\frac{\mu^3 u_1 (u-1)^3 (u+1)^3}{12} + \right. \\ \qquad \left. + \frac{\mu^2 u (u-1)(u+1) \left(u^2 - 2u_1^2 - 1 \right)}{12} - \frac{\mu u_1 \left(2u^2 + u_1^2 - 2 \right)}{12} - \frac{u}{12} \right) + \mathcal{O}(h^4) = 0. \end{cases} \tag{21}$$

Crank-Nicolson method FDA

$$\begin{cases} -u_1 + u_t + h^2 \left(-\frac{\mu^2 u_1 (u-1)^2 (u+1)^2}{12} - \frac{\mu u \left(u^2 - 2u_1^2 - 1 \right)}{12} + \frac{u_1}{12} \right) + \mathcal{O}(h^4) = 0, \\ \mu u_1 \left(u - 1 \right) \left(u + 1 \right) + u + u_{1\,t} + h^2 \left(\frac{\mu^3 u_1 (u-1)^3 (u+1)^3}{12} + \right. \\ \qquad \left. + \frac{\mu^2 u (u-1)(u+1) \left(u^2 - 8u_1^2 - 1 \right)}{12} - \frac{\mu u_1 \left(4u^2 - u_1^2 - 1 \right)}{6} - \frac{u}{12} \right) + \mathcal{O}(h^4) = 0. \end{cases} \tag{22}$$

Adams-Bashforth 4 step FDA

$$\begin{cases} -u_1 + u_t + h^4\left(\frac{251\mu^4 u_1(u-1)^4(u+1)^4}{720} + \ldots + \frac{251u_1}{720}\right) + \mathcal{O}(h^5) = 0, \\ \mu u_1\,(u-1)\,(u+1) + u + u_{1\,t} + \\ \qquad\qquad + h^4\left(-\frac{251\mu^5 u_1(u-1)^5(u+1)^5}{720} - \ldots - \frac{251u}{720}\right) + \mathcal{O}(h^5) = 0. \end{cases} \tag{23}$$

Adams-Moulton 4 step FDA

$$\begin{cases} -u_1 + u_t + h^4\left(-\frac{19\mu^4 u_1(u-1)^4(u+1)^4}{720} + \ldots - \frac{19u_1}{720}\right) + \mathcal{O}(h^5) = 0, \\ \mu u_1\,(u-1)\,(u+1) + u + u_{1\,t} + \\ \qquad\qquad + h^4\left(\frac{19\mu^5 u_1(u-1)^5(u+1)^5}{720} - \ldots + \frac{19u}{720}\right) + \mathcal{O}(h^5) = 0. \end{cases} \tag{24}$$

Adams-Bashforth 5 step FDA

$$\left\{\begin{array}{l} -u_1 + u_t + h^5\left(-\frac{95\mu^5 u_1(u-1)^5(u+1)^5}{288} + \ldots - \frac{95u}{288}\right) + \mathcal{O}(h^6) = 0, \\ \mu u_1\left(u-1\right)\left(u+1\right) + u + u_{1\,t} + \\ \qquad + h^5\left(\frac{95\mu^6 u_1(u-1)^6(u+1)^6}{288} - \ldots - \frac{95u_1}{288}\right) + \mathcal{O}(h^6) = 0. \end{array}\right. \tag{25}$$

Adams-Moulton 5 step FDA

$$\left\{\begin{array}{l} -u_1 + u_t + h^5\left(\frac{3\mu^5 u_1(u-1)^5(u+1)^5}{160} + \ldots + \frac{3u}{160}\right) + \mathcal{O}(h^6) = 0, \\ \mu u_1\left(u-1\right)\left(u+1\right) + u + u_{1\,t} + \\ \qquad + h^5\left(-\frac{3\mu^6 u_1(u-1)^6(u+1)^6}{160} - \ldots + \frac{3u_1}{160}\right) + \mathcal{O}(h^6) = 0. \end{array}\right. \tag{26}$$

Conclusion

The performed calculations show that usage of the FDA allows to estimate the discrepancy of the numerical methods with respect to the parameters of the problem, to detect and to evaluate the stiffness of the ODE system.

Furthermore, we can use residual FDA member for selection of variable step.

The source texts of the programs and the presented calculations are given at `github.com/blinkovua/sharing-blinkov/tree/master/FDA_ODE`. The presented method use a standard tools of computer algebra and can be applied systems with a polynomial right-had side.

[1] *Yanenko N.N., Shokin Yu.I.* The first differential approximation of difference schemes for hyperbolic systems of equations. Siberian Mathematical Journal. 1969. Vol. 10, N. 5. P. 868–880. DOI: 10.1007/BF00971662

[2] *Blinkov Y.A., Gerdt V.P., Marinov K.B.* Discretization of quasilinear evolution equations by computer algebra methods. Programming and Computer Software. 2017. Vol. 43, N. 2. P. 84–89. DOI: 10.1134/S0361768817020049

[3] *Blinkov Y.A., Malykh M.D., Sevastianov L.A.* On differential approximations of difference schemes. Izvestiya of Saratov University. Mathematics. Mechanics. Informatics. 2021. Vol. 21, N. 4. P. 472–488. DOI: 10.18500/1816-9791-2021-21-4-472-488

[4] Blinkov Y.A., Rebrina A.Y. Investigation of Difference Schemes for Two-Dimensional Navier-Stokes Equations by Using Computer Algebra Algorithms. Programming and Computer Software, 2023, vol. 49, pp. 26-31. DOI: 10.1134/S0361768823010024

[5] *Van der Pol B.* On “Relaxation-Oscillations”. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science. 1926. N. 2. P. 978–89992. DOI: 10.1080/14786442608564127

[6] *Kutta M.* Beitrag zur näherungsweisen Integration totaler Differentialgleichungen. Zeitschrift für Mathematik und Physik. 1901. Vol. 46. P. 435–453.

[7] *Curtiss C.F., Hirschfelder J.O.* Integration of stiff equations. Proceedings of the National Academy of Sciences of the USA. 1952. Vol. 38. N. 3. P. 235–243. DOI: 10.1073/pnas.38.3.235