Automatic confirmation of exhaustive use of information on a given equation

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Problem Statement

In several previous works we considered linear ordinary differential equations with coefficients given as truncated power series. We discussed the question of what can be learned from equations given in this way about their Laurent solutions, i.e. solutions belonging to the field of formal Laurent series.

- *Abramov S.A, Khmelnov D.E., Ryabenko, A.A.* Laurent solutions of linear ordinary differential equations with coefficients in the form of truncated power series. In: COMPUTER ALGEBRA, Moscow, June 17–21, 2019, International Conference Materials. pp. 75–82 (2019)

- *Abramov S.A, Khmelnov D.E., Ryabenko, A.A.* Linear Ordinary Differential Equations and Truncated Series. Computational Mathematics and Mathematical Physics, 2019. Vol. 59, N. 10, pp. 1649–1659.

- *Abramov S.A, Khmelnov D.E., Ryabenko, A.A.* Procedures for searching Laurent and regular solutions of linear differential equations with the coefficients in the form of truncated power series. Programming and Computer Software, 2020 Vol. 46, N. 2, pp. 67–75.

We were interested in the maximum possible information about these solutions, that is invariant with respect to possible prolongations of the truncated series which are the coefficients of the given equation.

Definition

A prolongation *of a truncated series is a series, possibly also truncated, whose initial terms coincide with the known initial terms of the original truncated series; correspondingly, the prolongation of an equation with truncated-series coefficients is an equation, whose coefficients are prolongations of the coefficients of the original equation.*

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The following is an example of the truncated equation (differential equations are written using the operator $\theta = x\frac{d}{dx}$ in the talk):

$$(x + O(x^2))\theta y(x) + (-x + O(x^2))y(x) = 0. \tag{1}$$

Two different prolongations of the equation (1) are, for example:

$$(x + O(x^3))\,\theta y(x) + \left(-x + \frac{x^3}{2} + O(x^4)\right) y(x) = 0 \tag{2}$$

$$\left(x + \frac{x^2}{2} + O(x^3)\right)\theta y(x) + \left(-x - x^2 - \frac{x^3}{2} + O(x^4)\right) y(x) = 0 \tag{3}$$

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The truncated Laurent solution of the equation (1) is

$$x\_c_1 + O(x^2).$$

The truncated Laurent solution of the equation (2) is

$$x\_c_1 + O(x^3).$$

The truncated Laurent solution of the equation (3) is

$$x\_c_1 + \frac{x^2\_c_1}{2} + O(x^3).$$

As we can see the truncated Laurent solution of the equation (1) is the maximal invariant one.

Algorithms for constructing such invariant truncated Laurent solutions and the implementation of the algorithms in Maple were presented in our previous works.

In other words, the presented algorithms provide exhaustive use of information on a given equation.

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Now we are focusing on the question of automatic confirmation of such exhaustive use of information on a given equation, i.e. the confirmation that it is not possible to add any additional terms to the constructed truncated solutions that are invariant with respect to prolongations of the given equation.

> In order to confirm this, it is sufficient to demonstrate a counterexample with two different prolongations of the given equation which lead to the appearance of different additional terms in the solutions.

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Implementation and Examples

The automatic confirmation of exhaustive use of the information on a given equation in the truncated Laurent solution is implemented as the Maple procedure `ExhaustiveUseConfirmation`.

It is based on finding Laurent solutions with *literals*, i.e., symbols used to represent the unspecified coefficients of the series involved in the equations. Those symbols are coefficients of the terms, the degrees of which are greater than the degree of the series truncation.

Finding Laurent solutions using literals means expressing the subsequent (not invariant to all possible prolongations) terms of the series in the solution as formulas in literals, i.e. via unspecified coefficients. This allows one to clarify the influence of unspecified coefficients on the subsequent terms of the series in the solutions.

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Example 1

Consider the following equation with the truncated-series coefficients and construct its Laurent solution using the TruncatedSeries package:

```
> eq := (-1+x+x^2+O(x^3))*theta(y(x),x,2)+
        (-2+O(x^3))*theta(y(x),x,1)+
        (x+6*x^2+O(x^4))*y(x);
```

$$eq := \left(-1 + x + x^2 + O(x^3)\right)\theta(y(x), x, 2) + \left(-2 + O(x^3)\right)\theta(y(x), x, 1) + \left(x + 6x^2 + O(x^4)\right)y(x)$$

```
> sol := TruncatedSeries:-LaurentSolution(eq,y(x));
```

$$sol := \left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x), \_c_2 + \frac{x\_c_2}{3} + \frac{5x^2\_c_2}{6} + \frac{13x^3\_c_2}{30} + O(x^4)\right]$$

The invocation of the procedure `ExhaustiveUseConfirmation` confirms exhaustive use of the information on the given equation with presenting two different prolongations of the equation that lead to two different prolongations of the solution.

The procedure prints out details on two different equation prolongations and their solutions. It is shown that the provided solutions are different prolongations of the solution of the given equation with presenting different additional terms in the solutions.

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```
> ExhaustiveUseConfirmation(sol, eq, y(x));
```

```
The equation prolongation #1
```

$$\left(-1+x+x^2-x^3+O(x^4)\right)\theta(y(x),x,2)+\left(-2-x^3+O(x^4)\right)\theta(y(x),x,1)+\left(x+6x^2-x^4+O(x^5)\right)y(x)$$

Additional term(s) in the equation prolongation:

$$y(x)(-x^4+O(x^5))+\theta(y(x),x,1)(-x^3+O(x^4))+\theta(y(x),x,2)(-x^3+O(x^4))$$

The equation solution:

$$\left[\frac{\_c_1}{x^2}-\frac{5\_c_1}{x}+\_c_2+x\left(\frac{\_c_2}{3}-\frac{37\_c_1}{3}\right)+O(x^2),\ \_c_2+\frac{x\_c_2}{3}+\frac{5x^2\_c_2}{6}+\frac{13x^3\_c_2}{30}+\frac{11x^4\_c_2}{24}+O(x^5)\right]$$

Additional term(s) in the equation solution:

$$\left[x\left(\frac{\_c_2}{3}-\frac{37\_c_1}{3}\right)+O(x^2),\ \frac{11x^4\_c_2}{24}+O(x^5)\right]$$

The equation prolongation #2

$$\left(-1+x+x^2+x^3+O(x^4)\right)\theta(y(x),x,2)+\left(-2+x^3+O(x^4)\right)\theta(y(x),x,1)$$
$$+\left(x+6x^2+x^4+O(x^5)\right)y(x)$$

Additional term(s) in the equation prolongation:

$$y(x)(x^4+O(x^5))+\theta(y(x),x,1)(x^3+O(x^4))+\theta(y(x),x,2)(x^3+O(x^4))$$

The equation solution:

$$\left[\frac{\_c_1}{x^2}-\frac{5\_c_1}{x}+\_c_2+x\left(\frac{\_c_2}{3}-11\_c_1\right)+O(x^2),\ \_c_2+\frac{x\_c_2}{3}+\frac{5x^2\_c_2}{6}\right.$$
$$\left.+\frac{13x^3\_c_2}{30}+\frac{43x^4\_c_2}{72}+O(x^5)\right]$$

Additional term(s) in the equation solution:

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Example 2

Consider a prolongation of the given equation with other additional terms (we use an auxiliary procedure `ConstructProlongation`) and construct its Laurent solution:

```
> eq1 := ConstructProlongation(theta(y(x),x,1)*x^3,eq,y(x))
```

$$\left(-1+x+x^2+O(x^3)\right)\theta(y(x),x,2)+\left(-2+x^3+O(x^4)\right)\theta(y(x),x,1) + \left(x+6x^2+O(x^4)\right)y(x)$$

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> TruncatedSeries:-LaurentSolution(eq1,y(x));
```

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We see that the solution is the same as the solution of the given equation eq.

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We see that the solution is the same as the solution of the given equation eq.

It shows that it is not sufficient just to construct the solutions of two random different prolongations for confirming exhaustive use of the information on a given equation. Supplementary information provided by the additional terms in a random prolongation does not necessarily lead to appearance of some additional terms in the equation solutions, so such a prolongation may not be used as a counterexample.

Example 3

Consider one more equation and its Laurent solution:

```
> eq := (x + O(x^2))*theta(y(x), x, 1) + O(x^2)*y(x);
```

$$eq := \left(x + O(x^2)\right)\theta(y(x), x, 1) + O(x^2)y(x)$$

```
> sol := TruncatedSeries:-LaurentSolution(eq,y(x));
```

$$sol := [\_c_1 + O(x)]$$

Instead of using `ExhaustiveUseConfirmation`, it is possible to check exhaustive use of the information on the given equation using two additional implemented procedures step by step. This way may be a better than using the text printed by `ExhaustiveUseConfirmation` when, for example, the details of the counterexample are needed in some further algorithmic processing.

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First, the invocation of the procedure `DifferentProlongationExtras` gives two different additional terms to construct two different prolongations of the given equation:

```
> dp := DifferentProlongationExtras(eq, y(x));
```

$$dp := \left[y(x)\left(-x^2 + O(x^3)\right), y(x)\left(x^2 + O(x^3)\right)\right]$$

Next, the procedure `ConstructProlongation` is applied twice to construct the equation prolongations.

```
> eq1 := ConstructProlongation(dp[1], eq, y(x));
```

$$eq1 := \left(x + O(x^2)\right)\theta(y(x), x, 1) + y(x)\left(-x^2 + O(x^3)\right)$$

```
> eq2 := ConstructProlongation(dp[2], eq, y(x));
```

$$eq2 := \left(x + O(x^2)\right)\theta(y(x), x, 1) + y(x)\left(x^2 + O(x^3)\right)$$

Finally, the Laurent solutions of each equation prolongation are constructed:

```
> sol1 := TruncatedSeries:-LaurentSolution(eq1, y(x))
```

$$sol1 := \left[\_c_1 + x\_c_1 + O(x^2)\right]$$

```
> sol2 := TruncatedSeries:-LaurentSolution(eq2, y(x))
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$$sol2 := \left[\_c_1 - x\_c_1 + O(x^2)\right]$$

We can see that the different equation prolongations lead to two different solution prolongations.

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