Laurent Solutions of Linear Ordinary Differential Equations with Coefficients in the Form of Truncated Power Series

S. Abramov, D. Khmelnov, A. Ryabenko

Dorodnicyn Comp. Centre, Federal Research Centre CSC of RAS,

The coefficients of a linear ordinary differential equation are quite often represented by series, and the problem can be in finding such solutions to this equation which are series of some fixed kind.

Below, the formal power series will be used as coefficients of a given equation, and, the coefficients of these series themselves, will be elements of a given differential field K of characteristic 0.

The solutions we are interested in, will be belonging to the field of formal Laurent series over K. Such solutions we will call *Laurent solutions*.

We will not be interested in the questions of convergence of series.

In this talk, we suppose that series playing the role of coefficients of a given equation are presented in an "approximate", namely, in *truncated* form.

Definition 1

Let K be a field of characteristic 0 and $K[x]$ the ring of polynomials with coefficients in K. We denote by $K[[x]]$ the ring of formal power series with coefficients in K and $K((x))$ its quotient field; the elements of $K((x))$ are *Laurent series*.

For a nonzero element $a(x) = \sum a_i x^i$ of $K((x))$ the *valuation* $\operatorname{val} a(x)$ is defined by $\operatorname{val} a(x) = \min \{i \mid a_i \neq 0\}$. By convention $\operatorname{val} 0 = \infty$.

Let $\ell \in \mathbb{Z} \cup \{-\infty\}$, the *$\ell$-truncation* $a^{\langle \ell \rangle}(x)$ is obtained by vanishing all the coefficients of the terms of degree larger than ℓ in the series $a(x)$; if $\ell = -\infty$ then $a^{\langle \ell \rangle}(x) = 0$.

Further K is a differential field of characteristic 0 with a differentiation $D = \frac{d}{dx}$. We will consider differential operators and equations written using the notation $\theta = x\frac{d}{dx}$. We assume that in the original operator

$$L = \sum_{i=0}^{r} a_i(x)\theta^{\,i} \in K[x][\theta], \tag{1}$$

coefficients are

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j,\ \ t_i \geq \deg a_i(x),\ \ \text{for } i = 0, 1, \ldots, r$$

(if $t_i > d_i = \deg a_i$ then $a_i = 0$ for $i = d_i + 1, d_i + 2, \ldots, t_i$). We assume that $a_r(x) \neq 0$ and the constant term of at least one of the polynomials $a_0(x), \ldots, a_r(x)$ is non-zero.

Definition 2

Let L have the form (1), the polynomial $a_r(x)$ (*the leading coefficient* of the differential operator L) is assumed to be nonzero. *A prolongation* of the operator L we will call any operator $\tilde{L} = \sum_{i=0}^{r} b_i(x)\theta^i \in K[[x]][\theta]$ for which

$$b_i(x) - a_i(x) = O(x^{t_i+1})$$

(i.e. $\mathrm{val}_x (b_i(x) - a_i(x)) > t_i$), $i = 0, 1, \ldots, r$.

So, a differential equation is factually given as

$$\left(a_r(x) + O(x^{t_r+1})\right) \theta^r y + \cdots + \left(a_1(x) + O(x^{t_1+1})\right) \theta y + \qquad (2)$$

$$+ \left(a_0(x) + O(x^{t_0+1})\right) y = 0,$$

$t_i \geq \deg a_i(x)$, $i = 0, 1, \ldots, r$. We associate the operator L with it, as well as the set of numbers $t_r, \ldots, t_0$.

A prolongation of the operator L will be called also a *prolongation of the equation* (2).

We are interested in the information on solutions belonging to the field of Laurent formal series which can be obtained from this representation of a given equation.

We emphasize that we are interested in such information about solutions which is invariant with respect to possible prolongations of the truncated series representing the coefficients of the equation.

Let σ denote the shift operator: $\sigma c_n = c_{n+1}$ for any sequence (c_n). The mapping

$$x \to \sigma^{-1}, \;\; \theta \to n$$

transforms the original differential equation $L(y) = 0$ into the *induced* recurrent equation

$$u_0(n)c_n + u_{-1}(n)c_{n-1} + \cdots = 0, \qquad (3)$$

where

- $(c_n)_{-\infty<n<\infty}$ is an unknown sequence such that $c_n = 0$ for all negative integers n with $|n|$ large enough.
- $u_{-j}(n) = \sum_{i=0}^{r} a_{i,j}\,(n-j)^i$ for $j = 0, 1, \ldots$

The constant term of at least one of $a_0(x), \ldots, a_r(x)$ is non-zero. This implies that $u_0(n)$ is a non-zero polynomial, but also that it does not depend on a prolongation of the original operator L.

The equation $L(y) = 0$ has a Laurent solution

$$y(x) = c_v x^v + c_{v+1} x^{v+1} + \dots$$

iff the double-sided sequence

$$\dots, 0,\ 0,\ c_v,\ c_{v+1},\ \dots$$

satisfies the induced recurrent equation, i.e.,

$$\begin{aligned}
&u_0(v)c_v = 0,\\
&u_0(v+1)c_{v+1} + u_{-1}(v+1)c_v = 0,\\
&u_0(v+2)c_{v+2} + u_{-1}(v+2)c_{v+1} + u_{-2}(v+2)c_v = 0,\\
&\dots.
\end{aligned}$$

The leading coefficient $u_0(n)$ can be considered as a kind of the *indicial polynomial* of the original differential equation. The set of the integer roots of $u_0(n)$ is finite and contains all possible valuations of Laurent solutions of the equation.

If $u_0(n) \neq 0$ for some integer n then the induced recurrent equation allows to find c_n by $c_{n-1}, c_{n-2}, \ldots$

If $u_0(n) = 0$ then we declare c_n as an *undetermined* coefficient. The preceding values of $c_{n-1}, c_{n-2}, \ldots$ should satisfy the relation

$$u_{-1}(n)c_{n-1} + u_{-2}(n)c_{n-2} + \cdots = 0. \tag{4}$$

Such relation has only finite number of non zero terms and will probably eliminate some of the previously undetermined coefficients.

Only after incrementing the value of n does this value exceed the largest integer root of the equation $u_0(n) = 0$, there is a guarantee that both new undetermined coefficients and relations of the form (4) will not arise already.

An algorithm will be proposed whose input is an operator $L \in K[x][\theta]$ and non-negative integers $t_0, t_1, \ldots, t_r$. A result of applying the algorithm is a finite set $Y = \{Y_{v_1,m_1}, \ldots, Y_{v_s,m_s}\}$ where

$$Y_{v_k,m_k} = \left\{ y^{\langle m_k \rangle}(x) : L(y) = 0, \operatorname{val} y^{\langle m_k \rangle}(x) = v_k \right\},$$

for $1 \leq k \leq s$, satisfies the following properties:

- If $y(x)$ is a solution of L such that $\operatorname{val}_x y(x) = v_k$, $1 \leq k \leq s$, then for any prolongation $\tilde{L}$ of the operator L there exists its solution $\tilde{y}(x)$, for which

$$\tilde{y}(x) = y^{\langle m_k \rangle}(x) + O(x^{m_k}) \text{ and } y^{\langle m_k \rangle}(x) \in Y_k. \tag{5}$$

- If $\tilde{y}(x)$ is a solution of some prolongation $\tilde{L}$ of L and $\operatorname{val} \tilde{y}(x) = v_k$, $1 \leq k \leq s$, then there is a solution $y(x)$ of the operator L, for which (5) holds.
- The values of m_k are the largest of the possible values associated with L in this way.

If the polynomial $u_0(n)$ does not have integer roots then none of the prolongations of the original differential equation has solutions in $K((x))$. The algorithm reports this and stops.

If there are integer roots $\alpha_1 < \cdots < \alpha_s$ then for L and its prolongations, only for α_s, the existence of a Laurent solution with such a value is guaranteed. We need to deal with $\alpha_1, \ldots, \alpha_{s-1}$. For each of these roots, there are three possibilities:

(a) Laurent solutions exist for all prolongations;

(b) Laurent solutions exist for some, but not for all prolongations;

(c) Laurent solutions do not exist for any prolongation.

By adding symbolic coefficients (not originally specified) to $a_i(x)$'s, it is possible to determine for each integer root α_k, which of the mentioned three possibilities takes place. If it is (a), then we find the corresponding m_k for

$$\tilde{y}(x) = y^{\langle m_k \rangle}(x) + O(x^{m_k+1}),$$

where $y(x)$ is a having the valuation α_k solution of the original truncated equation $L(y) = 0$, and $\tilde{y}(x)$ is a solution of a prolonged equation.
If it is (b) or (c), then we remove α_k from consideration.

Example 1

$$L = -\theta^2 - 2\,\theta, \quad t_0 = t_1 = t_2 = 0. \tag{6}$$

$u_0(n) = -n^2 - 2n$, its integer roots are $\alpha_1 = -2$, $\alpha_2 = 0$. $u_{-1}(n)$, $u_{-2}(n)$ contain unspecified coefficients of the prolongation of L.

Compute c_{-2}, c_{-1}, c_0 for $\alpha_1 = -2$:

- $n = -2$: $u_0(-2)\,c_{-2} = 0 \cdot c_{-2} = 0$, c_{-2} remains undetermined.
- $n = -1$: $u_0(-1)\,c_{-1} + u_{-1}(-1)\,c_{-2} = c_{-1} + u_{-1}(-2)\,c_{-2} = 0$. Thus $c_{-1} = -u_{-1}(-1)\,c_{-2}$.
- $n = 0$:
  $u_0(0)\,c_0 + u_{-1}(0)\,c_{-1} + u_{-2}(0)\,c_{-2} = 0\,c_0 + u_{-1}(0)\,c_{-1} + u_{-2}(0)\,c_{-2} = -u_{-1}(0)\,u_{-1}(-1)\,c_{-2} + u_{-2}(0)\,c_{-2} = (-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0))\,c_{-2} = 0$. c_0 remains undetermined with $c_{-2} = 0$, if $-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0) \neq 0$. Since $u_{-1}(n)$ and $u_{-2}(n)$ depend on the unspecified coefficients the case (b) is faced, i.e. Laurent solutions with the valuation $\alpha_1 = -2$ exist only if $-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0) = 0$. The valuation is removed.

For $\alpha_2 = 0$:

- $n = 0$: $u_0(0)\,c_0 = 0 \cdot c_0 = 0$, c_0 remains undetermined.
- $n = 1$: $u_0(1)\,c_1 + u_{-1}(1)\,c_0 = -3\,c_1 + u_{-1}(1)\,c_0 = 0$. Thus $c_1 = \frac{u_{-1}(1)\,c_0}{3}$. $u_{-1}(1)$, and hence c_1, depend on the unspecified coefficients.

  $n = 0$, corresponding to the maximal possible valuation is passed, so no further computation is needed.

The case (a) is faced here. Any prolongation of the equation has Laurent solution:

$$y(x) = C + O(x), \tag{7}$$

where C is an arbitrary constant, $C \neq 0$.

$$Y = \Big\{ Y_{0,0} = \big\{ C : C \in \mathbb{C} \setminus \{0\} \big\} \Big\}.$$

Example 2

Add new terms to the coefficients of L:

$$\tilde{L} = (-1 + x + x^2)\,\theta^2 - 2\,\theta + (x + 6x^2), \quad t_0 = 3, \; t_1 = t_2 = 2. \qquad (8)$$

$u_0(n)$ and its roots for $\tilde{L}$ are the same as for L.

$$u_{-1}(n) = (n-1)^2 + 1, \; u_{-2}(n) = (n-2)^2 + 6.$$

The computation for $\alpha_1 = -2$ is similar to the same for L. But for $n = 0$
$-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0) = -((0–1)^2 + 1)((-1–1)^2 + 1) + ((0–2)^2 + 6) = 0$,
and hence c_{-2} remains undetermined.
So $c_{-1} = -u_{-1}(-1)\,c_{-2} = -((-1-1)^2 + 1)\,c_{-2} = -5\,c_{-2}$.
$n = 0$, corresponding to the maximal possible valuation is passed and the expressions for further c_i contain unspecified coefficients, so no further computation is needed. The case (a) is faced here. Any prolongation of $\tilde{L}(y) = 0$ has Laurent solution with the valuation $v_1 = -2$:

$$Y_{-2,1} = \left\{ \frac{C_1}{x^2} - \frac{5\,C_1}{x} + C_2 : C_1 \in \mathbb{C} \setminus \{0\}, C_2 \in \mathbb{C} \right\}.$$

For $\alpha_2 = 0$:

- $n = 0$: $u_0(0)\, c_0 = 0 \cdot c_0 = 0$. c_0 remains undetermined.
- $n = 1$: $u_0(1)\, c_1 + u_{-1}(1)\, c_0 = -3\, c_1 + 1\, c_0 = 0$. Thus $c_1 = \frac{1}{3} c_0$.
- $n = 2$: $u_0(2)\, c_2 + u_{-1}(2)\, c_1 + u_{-2}(2)\, c_0 = -8\, c_2 + 2\, c_1 + 6\, c_0 = 0$. Thus $c_2 = \frac{5}{6} c_0$.
- $n = 3$: Compute $u_{-3}(n) = a(n-3)^2 + b(n-3)$, where a and b are unspecified coefficients. So $u_0(3)c_3 + u_{-1}(3)c_2 + u_{-2}(3)c_1 + u_{-3}(3)c_0$ $= -15\, c_3 + 5\, c_2 + 7\, c_1 + 0\, c_0 = 0$. Thus $c_3 = \frac{13}{30} c_0$. c_3 is computed, in spite of the fact that $u_{-3}(n)$ contains unspecified coefficients, since $u_{-3}(3) = 0$ for any values of these unspecified coefficients.

$n = 0$, corresponding to the maximal possible valuation is passed and the expressions for further c_i contain unspecified coefficients, so no further computation is needed. The case (a) is faced here as well. Any prolongation of $\tilde{L}(y) = 0$ has Laurent solution with the valuation $v_2 = 0$:

$$Y_{0,3} = \left\{ C + \frac{1}{3}\, Cx + \frac{5}{6}\, Cx^2 + \frac{13}{30}\, Cx^3 : C \in \mathbb{C} \setminus \{0\} \right\}.$$

Example 3

If new terms are added to the coefficients of (6) differently:

$$\tilde{\tilde{L}} = (-1 + x + x^2)\,\theta^2 + (-2 + x^2)\,\theta, \quad t_0 = 3,\ t_1 = t_2 = 2,$$

then the similar computation shows that

$$Y = \Big\{ Y_{0,3} = \{ C : C \in \mathbb{C} \setminus \{0\} \} \Big\}$$

for $\tilde{\tilde{L}}(y) = 0$. Any prolongation of the equation has Laurent solution: $C + O(x^4)$. The case (c) is faced here, which causes the removal of the possible valuation $\alpha_1 = -2$.