Symbolic-Numerical Investigation of Asymptotic Method for Studying Waveguide Propagation Problems

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5th International Conference "Computer Algebra"
Moscow, June 26-28, 2023

Introduction

The method for calculating waveguide modes propagating in irregular waveguides with slowly varying parameters was developed in works of Russian and foreign authors [1–8] and other works. In the Russian-language scientific literature, the most widely used methods are the "cross-section method" developed by B.Z.Katsenelenbaum [3, 4] for closed waveguides and its generalization for open waveguides developed by V.V. Shevchenko [5, 9].

The main difficulty in formulating the problem of finding guided waveguide modes in irregular waveguides—waveguides with a curvilinear boundary—is to take into account the conditions of continuity of the tangential field on a curvilinear boundary.

Introduction

The concept of guided mode is defined for planar waveguides – waveguides with flat boundaries. Applying the separation of variables to the waveguide problem for planar waveguides, we can formulate a self-adjoint spectral problem, whose eigenvalues are real phase retardation coefficients of the guided modes and the eigenfunctions determine the corresponding standing waves in the waveguide cross section.

In the case of a curvilinear boundary, the variables cannot be separated and the concept of a guided mode in an irregular waveguide is not strictly defined, although various approximate guided modes of an irregular waveguide can be constructed.

Introduction

As a rule, methods for the approximate construction of guided modes of a smoothly irregular waveguide are based on an approximation with respect to a small parameter characterizing a slow change in the geometry of the structure or the field along one of the coordinates.

In the Katsenelenbaum's and Shevchenko's approach, when constructing the approximate modes of an irregular waveguide, only the zeroth-order contribution to the field-matching conditions at the curvilinear boundary is taken into account: the variable thickness of the waveguide layer is taken into account, but the contributions of higher orders, which describe the small slope of the curvilinear interface, are discarded.

Introduction

Katsenelenbaum's and Shevchenko's approaches makes it possible to obtain guided modes slowly varying along the waveguide, and the guided modes obtained in each cross section of the irregular waveguide coincide with the guided modes of a regular waveguide of the same thickness. In each cross section, the spectral problem is self-adjoint; therefore, the phase deceleration coefficients are real slowly varying functions.

However, the contributions of the first and higher orders of smallness in terms of the slope of the boundary in radians are not taken into account in the cross-section method.

Introduction

The authors of this work are aware of an alternative approach to constructing waveguide modes in the adiabatic approximation, which takes into account the contributions of higher orders of smallness that describe the slope of the curvilinear interface.

The model of adiabatic waveguide modes (AWM) [10–12] is based on the short-wave approximation adapted for waveguide propagation, described in the book by V.M. Babich and V.S. Buldyrev [13].

Motivation for the use of CAS

The convenience of asymptotic methods is that the zeroth approximation, which describes the main contribution of the solution, is usually found in symbolic form.

Given the asymptotic character of the solution in the AWM model, it is possible to obtain a number of intermediate results in symbolic form, which predetermines the use of computer algebra as one of the research tools.

AWM model

In [14], the zeroth contribution was obtained to the adiabatic approximation of the waveguide solution of Maxwell's equations of the form

$$\left\{ \begin{array}{c} \vec{E}\left(x,y,z,t\right) \\ \vec{H}\left(x,y,z,t\right) \end{array} \right\} = \left\{ \begin{array}{c} \vec{E}_0\left(x;y,z\right) \\ \vec{H}_0\left(x;y,z\right) \end{array} \right\} \exp\left\{i\omega t - ik_0\varphi\left(y,z\right)\right\}, \qquad (1)$$

where

$$\varepsilon\frac{\partial E_0^y}{\partial x} = -ik_0\left(\frac{\partial\varphi}{\partial y}\right)\left(\frac{\partial\varphi}{\partial z}\right)H_0^y - ik_0\left(\varepsilon\mu - \left(\frac{\partial\varphi}{\partial y}\right)^2\right)H_0^z, \tag{2}$$

$$\varepsilon\frac{\partial E_0^z}{\partial x} = ik_0\left(\varepsilon\mu - \left(\frac{\partial\varphi}{\partial z}\right)^2\right)H_0^y + ik_0\left(\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial\varphi}{\partial y}\right)H_0^z, \tag{3}$$

$$\mu\frac{\partial H_0^y}{\partial x} = ik_0\left(\frac{\partial\varphi}{\partial y}\right)\left(\frac{\partial\varphi}{\partial z}\right)E_0^y + ik_0\left(\varepsilon\mu - \left(\frac{\partial\varphi}{\partial y}\right)^2\right)E_0^z, \tag{4}$$

$$\mu\frac{\partial H_0^z}{\partial x} = -ik_0\left(\varepsilon\mu - \left(\frac{\partial\varphi}{\partial z}\right)^2\right)E_0^y - ik_0\left(\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial\varphi}{\partial y}\right)E_0^z, \tag{5}$$

$$E_0^x = -\frac{\partial\varphi}{\partial y}\frac{1}{\varepsilon}H_0^z + \frac{\partial\varphi}{\partial z}\frac{1}{\varepsilon}H_0^y, \tag{6}$$

$$H_0^x = \frac{\partial\varphi}{\partial y}\frac{1}{\mu}E_0^z - \frac{\partial\varphi}{\partial z}\frac{1}{\mu}E_0^y. \tag{7}$$

AWM model

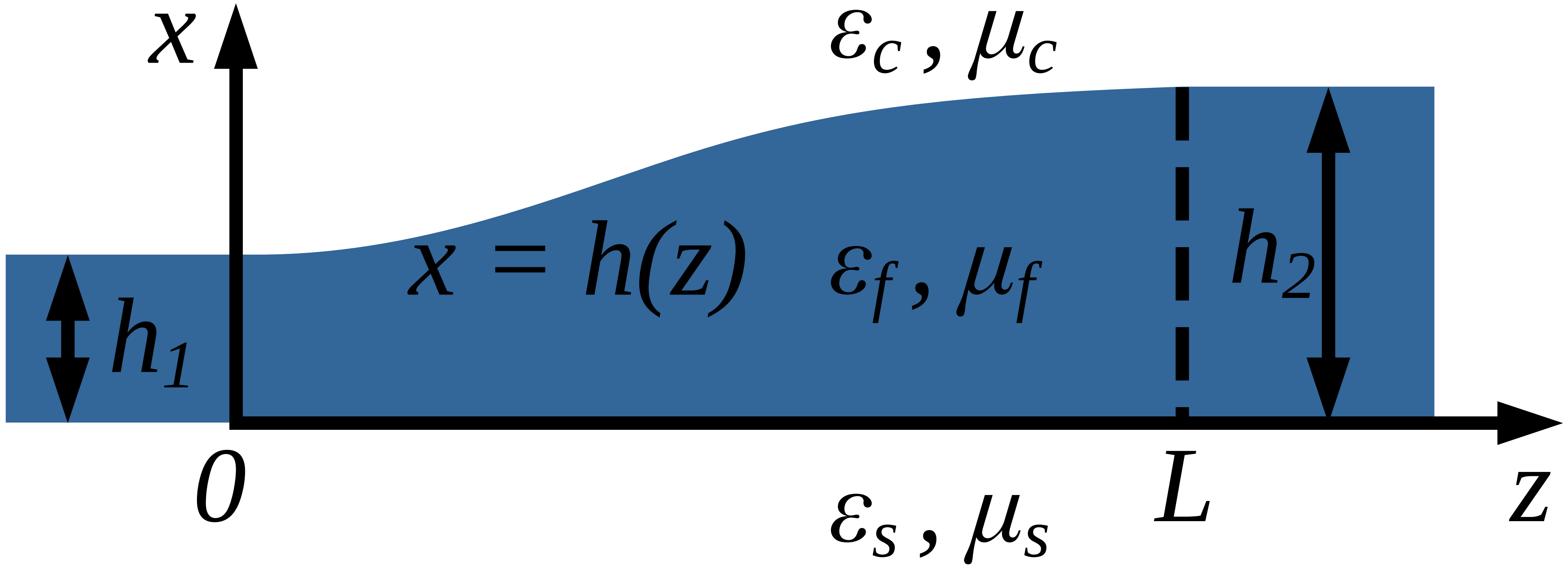
For a thin-film multilayer waveguide consisting of optically homogeneous layers, the system of equations (2)–(7) should be supplemented with the conditions for the matching of the electromagnetic field at the interfaces between the media [16]. At the interfaces of dielectric media, the following boundary conditions are satisfied:

$$\left[n \times \vec{E}\right]_{x=h(y,z)} = \vec{0}, \quad \left[n \times \vec{H}\right]_{x=h(y,z)} = \vec{0}, \tag{8}$$

In addition, we have the asymptotic boundary conditions at infinity [16]:

$$\left\|\vec{E}\right\| \xrightarrow[x\to\pm\infty]{} 0, \left\|\vec{H}\right\| \xrightarrow[x\to\pm\infty]{} 0. \tag{9}$$

Geometry of a two-dimensional smoothly irregular waveguide transition



AWM model

We consider the case when neither the geometry of the integrated optical waveguide nor the solutions of Maxwell's equations for the AWM depend on y. Then, Eqs. (2)–(5) take a simpler form:

$$\begin{cases} \varepsilon \dfrac{\partial E_0^y}{\partial x} = -ik_0 \varepsilon \mu H_0^z, \\ \varepsilon \dfrac{\partial E_0^z}{\partial x} = ik_0 \left(\varepsilon \mu - \left(\varphi' \left(z \right) \right)^2 \right) H_0^y, \\ \mu \dfrac{\partial H_0^y}{\partial x} = ik_0 \varepsilon \mu E_0^z, \\ \mu \dfrac{\partial H_0^z}{\partial x} = -ik_0 \left(\varepsilon \mu - \left(\varphi' \left(z \right) \right)^2 \right) E_0^y, \end{cases} \tag{10}$$

AWM model

and additional conditions (6) and (7) are also simplified:

$$E_0^x = \frac{1}{\varepsilon}\varphi'(z)\,H_0^y,\ \ H_0^x = -\frac{1}{\mu}\varphi'(z)\,E_0^y. \tag{11}$$

At a point of the interface , the fieldmatching conditions comprise the continuity of the following quantities:

$$\left[\vec{n}\times\vec{E}\right] = \left(h'(z)\cdot E_y;\ -E_z - h'(z)\cdot E_x;\ E_y\right)^T, \tag{12}$$

$$\left[\vec{n}\times\vec{H}\right] = \left(h'(z)\cdot H_y;\ -H_z - h'(z)\cdot H_x;\ H_y\right)^T, \tag{13}$$

AWM model

The waveguide parameters are as follows:

$\mu_c = \mu_f = \mu_s = 1$, $\varepsilon_c = 1$, $\varepsilon_f = 1.565^2$, $\varepsilon_s = 1.47^2$

the thicknesses are defined as $h_1 = 2\lambda$ and $h_2 = 3\lambda$; and $L = 100\lambda$, where λ is the wavelength, $\lambda = 0.55$ µm. The variable thickness is defined as follows:

$$h\left(z\right) = 2\left(h_1 - h_2\right)\left(\frac{z}{L}\right)^3 - 3\left(h_1 - h_2\right)\left(\frac{z}{L}\right)^2 + h_1, \qquad (14)$$

Symbolic method for solving the problem

We consider system (10). System (10) is formulated in symbolic form; the coefficients ε and μ of the system for the considered waveguide are piecewise constant functions; therefore, in each region of their constancy, we solve system (10) symbolically in the `Maple` computer algebra system [17]. As a result we obtain an expansion of the solution in the fundamental system of solutions (FSS) with undetermined coefficients.

Symbolic method for solving the problem

$$solution := \begin{bmatrix} -\mathrm{I}\, C2\, \mathrm{e}^{\eta(z)\, k x}\, \mu\, \eta(z) + \mathrm{I}\, C4\, \mathrm{e}^{-\eta(z)\, k x}\, \mu\, \eta(z) \\ C2\, \mathrm{e}^{\eta(z)\, k x}\, p(z) + C4\, \mathrm{e}^{-\eta(z)\, k x}\, p(z) \\ \mathrm{I}\, C1\, \mathrm{e}^{\eta(z)\, k x}\, \epsilon\, \eta(z) - \mathrm{I}\, C3\, \mathrm{e}^{-\eta(z)\, k x}\, \epsilon\, \eta(z) \\ C1\, \mathrm{e}^{\eta(z)\, k x}\, p(z) + C3\, \mathrm{e}^{-\eta(z)\, k x}\, p(z) \end{bmatrix}$$

$$subs\_symbolic\_eta := \left[\eta_c(z) = \sqrt{\phi^2 - \epsilon_c\, \mu_c},\, \eta_f(z) = \sqrt{\phi^2 - \epsilon_f \mu_f},\, \eta_s(z) = \sqrt{\phi^2 - \epsilon_s\, \mu_s}\right]$$

$$subs\_symbolic\_p := \left[p_c(z) = \phi^2 - \epsilon_c\, \mu_c,\, p_f(z) = \phi^2 - \epsilon_f \mu_f,\, p_s(z) = \phi^2 - \epsilon_s\, \mu_s\right]$$

Symbolic method for solving the problem

Solutions in semiinfinite layers should not increase at infinity according to conditions (9), due to which the constants multiplying the FSS functions increasing at infinity will be determined and equal to zero.

Writing the continuity conditions (12) and (13) at the layers' boundaries in the `Maple` computer algebra system [17], we obtain a homogeneous system of algebraic equations $M\vec{A} = \vec{0}$.

Symbolic method for solving the problem

$$M = \begin{pmatrix} \multicolumn{4}{c}{\multirow{4}{*}{M_1}} & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \multicolumn{4}{c}{\multirow{4}{*}{M_2}} \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{pmatrix} \qquad \vec{A} = \begin{pmatrix} A_c \\ A_f \\ C_f \\ A_s \\ B_c \\ D_f \\ B_f \\ B_s \end{pmatrix} \Rightarrow \begin{cases} M_1\vec{A}_1 = \vec{0}, \\ M_2\vec{A}_2 = \vec{0}. \end{cases}$$

$$\vec{A}_1 = \begin{pmatrix} A_c \\ A_f \\ C_f \\ A_s \end{pmatrix} \qquad \vec{A}_2 = \begin{pmatrix} B_c \\ D_f \\ B_f \\ B_s \end{pmatrix}$$

Symbolic method for solving the problem

$$M1 := \begin{bmatrix} I\,h'\eta_c(z)\,\phi' - \eta_c(z)^2 & I\,h'\eta_f(z)\,e^{\eta_f(z)\,k\,h}\,\phi' + e^{\eta_f(z)\,k\,h}\,\eta_f(z)^2 & -I\,h'\phi'\eta_f(z)\,e^{-\eta_f(z)\,k\,h} + e^{-\eta_f(z)\,k\,h}\,\eta_f(z)^2 & 0 \\ 0 & I\,\eta_f(z)\,\epsilon_f & -I\,\eta_f(z)\,\epsilon_f & -I\,\eta_s(z)\,\epsilon_s \\ 0 & -\eta_f(z)^2 & -\eta_f(z)^2 & \eta_s(z)^2 \\ -I\,\eta_c(z)\,\epsilon_c & -I\,e^{\eta_f(z)\,k\,h}\,\eta_f(z)\,\epsilon_f & I\,\eta_f(z)\,e^{-\eta_f(z)\,k\,h}\,\epsilon_f & 0 \end{bmatrix}$$

$$M2 := \begin{bmatrix} 0 & -\eta_f(z)^2 & -\eta_f(z)^2 & \eta_s(z)^2 \\ 0 & I\,\eta_f(z)\,\mu_f & -I\,\eta_f(z)\,\mu_f & I\,\eta_s(z)\,\mu_s \\ I\,h'\eta_c(z)\,\phi' - \eta_c(z)^2 & -I\,h'\phi'\eta_f(z)\,e^{-\eta_f(z)\,k\,h} + e^{-\eta_f(z)\,k\,h}\,\eta_f(z)^2 & I\,h'\eta_f(z)\,e^{\eta_f(z)\,k\,h}\,\phi' + e^{\eta_f(z)\,k\,h}\,\eta_f(z)^2 & 0 \\ I\,\eta_c(z)\,\mu_c & -I\,\eta_f(z)\,e^{-\eta_f(z)\,k\,h}\,\mu_f & I\,e^{\eta_f(z)\,k\,h}\,\eta_f(z)\,\mu_f & 0 \end{bmatrix}$$

Symbolic method for solving the problem

Let us apply the previously developed method [15] for solving in symbolic form homogeneous system of algebraic equations $M\vec{A} = \vec{0}$.

Symbolic solution of SLAE

$$A_c = \frac{\eta_f(z)^2 \left(\left(I\,\eta_f(z) + h'\,\phi' \right) \left(\eta_s(z)\,\epsilon_f - \eta_f(z)\,\epsilon_s \right) e^{-\eta_f(z)\,k\,h} + \left(\eta_f(z)\,\epsilon_s + \eta_s(z)\,\epsilon_f \right) e^{\eta_f(z)\,k\,h} \left(I\,\eta_f(z) - h'\,\phi' \right) \right) \eta_s(z)}{\eta_c(z) \left(I\,h'\,\phi' - \eta_c(z) \right)}$$

$$A_f = -I\,\eta_f(z)\,\eta_s(z) \left(\eta_f(z)\,\epsilon_s + \eta_s(z)\,\epsilon_f \right)$$

$$A_s = -2\,I\,\eta_f(z)^3\,\epsilon_f$$

$$C_f = I\,\eta_f(z)\,\eta_s(z) \left(\eta_f(z)\,\epsilon_s - \eta_s(z)\,\epsilon_f \right)$$

$$B_c = \frac{-I\,\eta_f(z)^2\,\eta_s(z)\,\mu_f \left(\eta_f(z)\,e^{\eta_f(z)\,k\,h}\,\mu_s + \eta_f(z)\,e^{-\eta_f(z)\,k\,h}\,\mu_s + e^{\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f - e^{-\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f \right)}{\mu_c\,\eta_c(z)}$$

$$B_f = I\,\eta_f(z)\,\eta_s(z) \left(\eta_f(z)\,\mu_s + \eta_s(z)\,\mu_f \right)$$

$$B_s = 2\,I\,\eta_f(z)^3\,\mu_f$$

$$D_f = I\,\eta_f(z)\,\eta_s(z) \left(\eta_s(z)\,\mu_f - \eta_f(z)\,\mu_s \right)$$

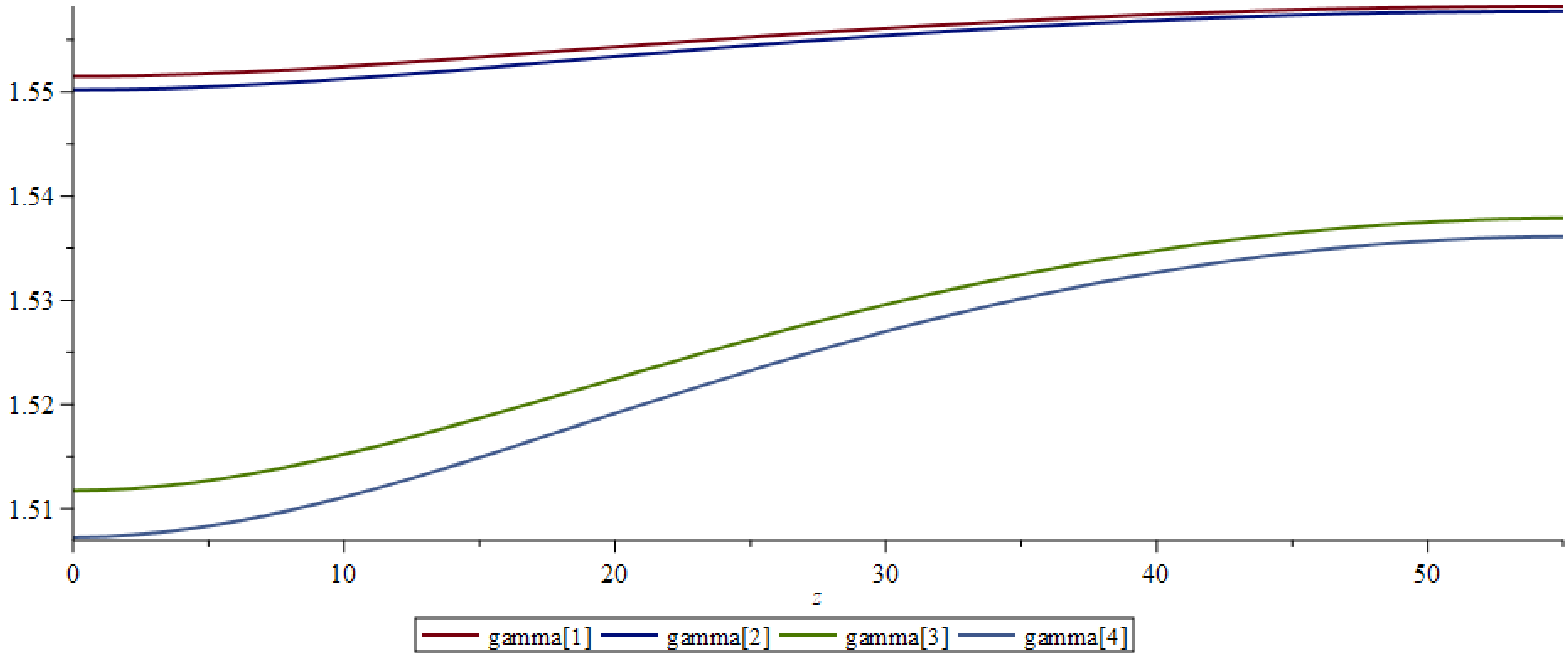
Numeric solution of nonlinear equation

The condition for the solvability of the system $M\vec{A}=\vec{0}$ is the equality to zero of the determinant of the system, which is a nonlinear equation: $\det M\big(z,\gamma(z)\big)=0$ where $\gamma(z)=\varphi'(z)$, $\varphi(z)$ – the phase deceleration coefficient.

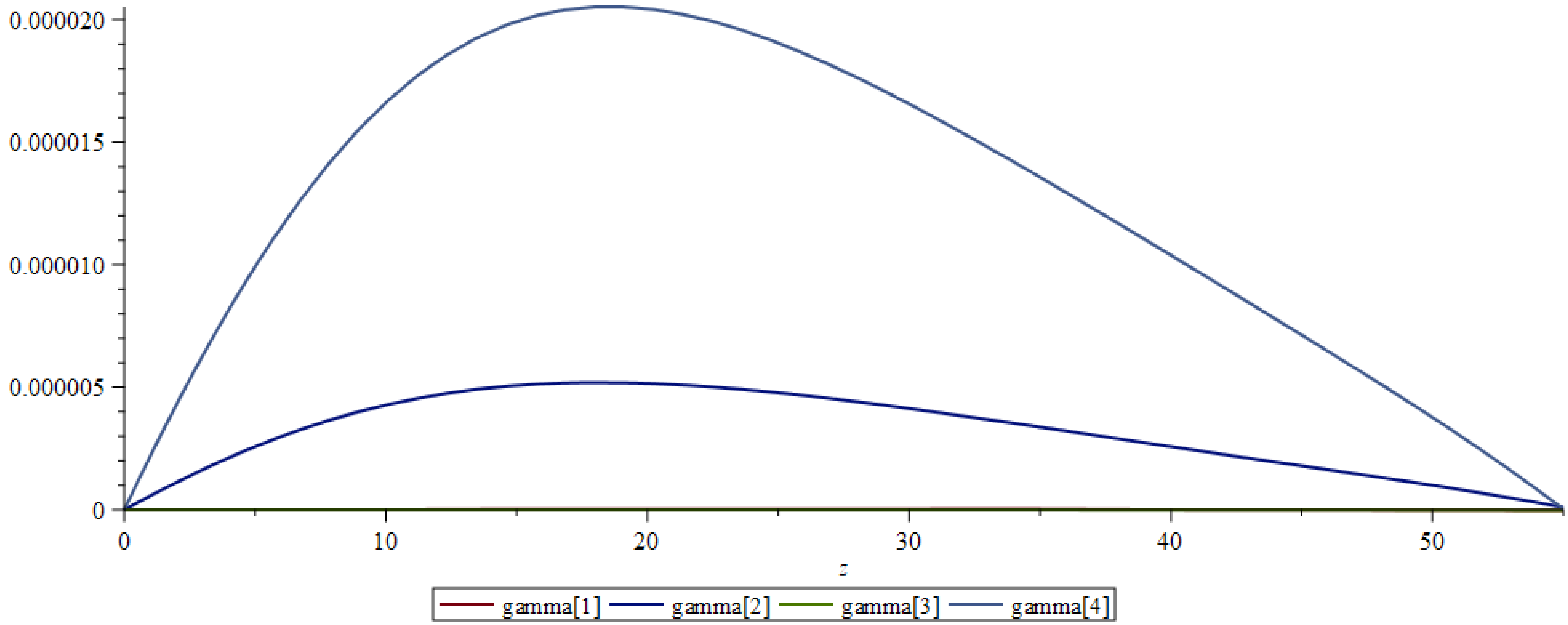
We apply the so called “parameter continuation method” to solve nonlinear equation $F\big(z,\gamma(z)\big)=0$. We differentiate the equation symbolically with respect to z: $\frac{\partial F}{\partial z}+\frac{\partial F}{\partial \gamma}\cdot\gamma'=0$ and formulate the Cauchy problem

$$\gamma'=-\frac{\partial F/\partial z}{\partial F/\partial \gamma}$$
$$\gamma(z_0)=\gamma_0$$

Real part of $\gamma(z)$



Imaginary part of $\gamma(z)$



Electromagnetic field for covering layer

$$\begin{bmatrix} Ex \\ Ey \\ Ez \\ Hx \\ Hy \\ Hz \end{bmatrix} = \begin{bmatrix} \dfrac{-I\,\eta_f(z)^2\left(\left(I\,\eta_f(z)+h_z\,\phi'\right)\left(\eta_s(z)\,\epsilon_f-\eta_f(z)\,\epsilon_s\right)e^{-\eta_f(z)\,k\,h}+\left(\eta_f(z)\,\epsilon_s+\eta_s(z)\,\epsilon_f\right)e^{\eta_f(z)\,k\,h}\left(I\,\eta_f(z)-h_z\,\phi'\right)\right)\eta_s(z)\,e^{-\eta_c(z)\,k\,(x-h)}\,\phi'}{I\,h_z\,\phi'-\eta_c(z)} \\ \eta_f(z)^2\,\eta_s(z)\,\mu_f\left(\eta_f(z)\,e^{\eta_f(z)\,k\,h}\,\mu_s+\eta_f(z)\,e^{-\eta_f(z)\,k\,h}\,\mu_s+e^{\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f-e^{-\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f\right)e^{-\eta_c(z)\,k\,(x-h)} \\ \dfrac{\eta_f(z)^2\left(\left(I\,\eta_f(z)+h_z\,\phi'\right)\left(\eta_s(z)\,\epsilon_f-\eta_f(z)\,\epsilon_s\right)e^{-\eta_f(z)\,k\,h}+\left(\eta_f(z)\,\epsilon_s+\eta_s(z)\,\epsilon_f\right)e^{\eta_f(z)\,k\,h}\left(I\,\eta_f(z)-h_z\,\phi'\right)\right)\eta_s(z)\,\eta_c(z)\,e^{-\eta_c(z)\,k\,(x-h)}}{I\,h_z\,\phi'-\eta_c(z)} \\ -\dfrac{\eta_f(z)^2\,\eta_s(z)\,\mu_f\left(\eta_f(z)\,e^{\eta_f(z)\,k\,h}\,\mu_s+\eta_f(z)\,e^{-\eta_f(z)\,k\,h}\,\mu_s+e^{\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f-e^{-\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f\right)e^{-\eta_c(z)\,k\,(x-h)}\,\phi'}{\mu_c} \\ \dfrac{-I\,\eta_f(z)^2\left(\left(I\,\eta_f(z)+h_z\,\phi'\right)\left(\eta_s(z)\,\epsilon_f-\eta_f(z)\,\epsilon_s\right)e^{-\eta_f(z)\,k\,h}+\left(\eta_f(z)\,\epsilon_s+\eta_s(z)\,\epsilon_f\right)e^{\eta_f(z)\,k\,h}\left(I\,\eta_f(z)-h_z\,\phi'\right)\right)\eta_s(z)\,e^{-\eta_c(z)\,k\,(x-h)}\,\epsilon_c}{I\,h_z\,\phi'-\eta_c(z)} \\ \dfrac{-I\,\eta_f(z)^2\,\eta_s(z)\,\mu_f\left(\eta_f(z)\,e^{\eta_f(z)\,k\,h}\,\mu_s+\eta_f(z)\,e^{-\eta_f(z)\,k\,h}\,\mu_s+e^{\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f-e^{-\eta_f(z)\,k\,h}\,\eta_s(z)\,\mu_f\right)\eta_c(z)\,e^{-\eta_c(z)\,k\,(x-h)}}{\mu_c} \end{bmatrix}$$

Electromagnetic field for waveguide layer

$$\begin{bmatrix} Ex \\ Ey \\ Ez \\ Hx \\ Hy \\ Hz \end{bmatrix} = \begin{bmatrix} I\,\eta_f(z)\left(-I\,\eta_f(z)\,\eta_s(z)\left(\eta_f(z)\,\epsilon_s+\eta_s(z)\,\epsilon_f\right)e^{\eta_f(z)\,kx}-I\,\eta_f(z)\,\eta_s(z)\left(\eta_f(z)\,\epsilon_s-\eta_s(z)\,\epsilon_f\right)e^{-\eta_f(z)\,kx}\right)\phi' \\ \eta_f(z)^2\,\eta_s(z)\left(\eta_f(z)\,\mu_s+\eta_s(z)\,\mu_f\right)e^{\eta_f(z)\,kx}\,\mu_f-\eta_f(z)^2\,\eta_s(z)\left(\eta_s(z)\,\mu_f-\eta_f(z)\,\mu_s\right)e^{-\eta_f(z)\,kx}\,\mu_f \\ -I\,\eta_f(z)^3\,\eta_s(z)\left(\eta_f(z)\,\epsilon_s+\eta_s(z)\,\epsilon_f\right)e^{\eta_f(z)\,kx}+I\,\eta_f(z)^3\,\eta_s(z)\left(\eta_f(z)\,\epsilon_s-\eta_s(z)\,\epsilon_f\right)e^{-\eta_f(z)\,kx} \\ I\,\eta_f(z)\left(I\,\eta_f(z)\,\eta_s(z)\left(\eta_f(z)\,\mu_s+\eta_s(z)\,\mu_f\right)e^{\eta_f(z)\,kx}-I\,\eta_f(z)\,\eta_s(z)\left(\eta_s(z)\,\mu_f-\eta_f(z)\,\mu_s\right)e^{-\eta_f(z)\,kx}\right)\phi' \\ \eta_f(z)^2\,\eta_s(z)\left(\eta_f(z)\,\epsilon_s+\eta_s(z)\,\epsilon_f\right)e^{\eta_f(z)\,kx}\,\epsilon_f+\eta_f(z)^2\,\eta_s(z)\left(\eta_f(z)\,\epsilon_s-\eta_s(z)\,\epsilon_f\right)e^{-\eta_f(z)\,kx}\,\epsilon_f \\ I\,\eta_f(z)^3\,\eta_s(z)\left(\eta_f(z)\,\mu_s+\eta_s(z)\,\mu_f\right)e^{\eta_f(z)\,kx}+I\,\eta_f(z)^3\,\eta_s(z)\left(\eta_s(z)\,\mu_f-\eta_f(z)\,\mu_s\right)e^{-\eta_f(z)\,kx} \end{bmatrix}$$

Electromagnetic field for substrate layer

$$\begin{bmatrix} Ex \\ Ey \\ Ez \\ Hx \\ Hy \\ Hz \end{bmatrix} = \begin{bmatrix} 2\,\eta_s(z)\,\eta_f(z)^3\,\epsilon_f \mathrm{e}^{\eta_s(z)\,kx}\,\phi' \\ 2\,\eta_f(z)^3\,\mu_f \mathrm{e}^{\eta_s(z)\,kx}\,\mu_s\,\eta_s(z) \\ -2\,\mathrm{I}\,\eta_f(z)^3\,\epsilon_f \mathrm{e}^{\eta_s(z)\,kx}\,\eta_s(z)^2 \\ -2\,\eta_s(z)\,\eta_f(z)^3\,\mu_f \mathrm{e}^{\eta_s(z)\,kx}\,\phi' \\ 2\,\eta_f(z)^3\,\epsilon_f \mathrm{e}^{\eta_s(z)\,kx}\,\epsilon_s\,\eta_s(z) \\ 2\,\mathrm{I}\,\eta_f(z)^3\,\mu_f \mathrm{e}^{\eta_s(z)\,kx}\,\eta_s(z)^2 \end{bmatrix}$$

Results

- Due to the presence of a symbolic solution of the considered system (10) of ordinary differential equations, it is possible to formulate the problem of searching for waveguide modes in symbolic form.
- The system of linear equations obtained in symbolic form can be simplified, namely, represented as two independent subsystems of smaller dimensions.
- By means of computer algebra, the obtained system is solved in symbolic form in the computer algebra system `Maple`.
- The nonlinear equation describing the solvability condition of the system is solved numerically in the computer algebra system `Maple`.
- The electromagnetic field of waveguide modes is defined in symbolic-numerical form.

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