# Integrable Cases of the Euler—Poisson Equations


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**COMPUTER ALGEBRA, June 23–25, 2025**

# Talk outlook

1. Introduction
2. Theory
3. Algorithm of searching for integrable cases
4. Results for case $\ell = 1$
5. References

## Abstract

In the classical problem of the motion of a rigid body around a fixed point, described by the Euler-Poisson equations, we propose a new method for computing cases of integrability: first, we provide algorithms for computing values of parameters ensuring potential integrability and then we select cases of global integrability. By the method we have obtained all the known cases of global integrability, six new cases of potential integrability, for which the absence of their global integrability are proven.

1. **Introduction**
2. Theory
3. Algorithm of searching for integrable cases
4. Results for case $\ell = 1$
5. References

This study introduces an innovative approach to identify integrable cases of motion of the rigid top with a fixed point. Using the structural characteristics of the dynamical equations and developing new invariants within a unified framework, we seek to advance previous methodologies and expand the repertoire of recognized integrable configurations.

The system of ***Euler-Poisson equations*** (1750) (or shortly ***EP-equations***) is a real autonomous system of six ordinary differential equations (ODEs).

$$\begin{aligned}
&Ap' + (C - B)qr = Mg\,(y_0\gamma_3 - z_0\gamma_2)\,,\\
&Bq' + (A - C)pr = Mg\,(z_0\gamma_1 - x_0\gamma_3)\,,\\
&Cr' + (B - A)pq = Mg\,(x_0\gamma_2 - y_0\gamma_1)\,,\\
&\gamma_1' = r\gamma_2 - q\gamma_3, \quad \gamma_2' = p\gamma_3 - r\gamma_1, \quad \gamma_3' = q\gamma_1 - p\gamma_2,
\end{aligned} \tag{1}$$

with dependent variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$ and parameters $A, B, C$, $x_0, y_0, z_0$, satisfying the triangle inequalities

$$0 < A \leqslant B + C, \quad 0 < B \leqslant A + C, \quad 0 < C \leqslant A + B. \tag{2}$$

Here, the prime indicates differentiation over the independent variable time $t$, $Mg$ is the weight of the body, $A, B, C$ are the principal moments of inertia of the rigid body, $x_0, y_0, z_0$ are the coordinates of the center of mass of the rigid body, $\gamma_1, \gamma_2, \gamma_3$ are the vertical directional cosines.

EP-equations describe the motion of a rigid top around a fixed point [Golubev, 1960] and have the following three first integrals:

$$\textit{\textbf{energy}}: \ I_1 \stackrel{\text{def}}{=} Ap^2 + Bq^2 + Cr^2 - 2Mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \text{const},$$

$$\textit{\textbf{momentum}}: \ I_2 \stackrel{\text{def}}{=} Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const},$$

$$\textit{\textbf{geometric}}: \ I_3 \stackrel{\text{def}}{=} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

EP-equations are integrable if there is a fourth general integral $I_4$. So far, 5 cases of integrability are known:

Case 1. *Euler-Poinsot*: $x_0 = y_0 = z_0 = 0$ and the fourth integral is

$$I_4 \stackrel{\text{def}}{=} A^2p^2 + B^2q^2 + C^2r^2 = \text{const}.$$

Case 2. *Lagrange-Poisson*: $B = C$, $x_0 \neq 0$, $y_0 = z_0 = 0$, and the fourth integral is

$$I_4 \stackrel{\text{def}}{=} p = \text{const}.$$

Case 3. *Kovalevskaya* **(1890)**: $A = B = 2C$, $x_0 \neq 0$, $y_0 = z_0 = 0$, and

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1\right)^2 + (2pq + c\gamma_2)^2 = \text{const},$$

where $c = Mgx_0/C$.

Case 4. *Kinematic symmetry*: $A = B = C$ and $I_4 \stackrel{\text{def}}{=} x_0p + y_0q + z_0r = \text{const}$. It is derived from Case 2.

**Case 5.** ***Bruno–Batkhin*** **(2024) [Bruno, Batkhin, 2024]:** $A = B = 2C$, $x_0 \neq 0$, $y_0 \neq 0$, $z_0 = 0$, the fourth integral is

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1 - d\gamma_2\right)^2 + (2pq + d\gamma_1 + c\gamma_2)^2 = \text{const},$$

where $c = Mgx_0/C$, $d = Mgy_0/C$.

Below $\ell$ is the number of nonzero values among parameters $x_0, y_0, z_0$.

1. Introduction

2. Theory

3. Algorithm of searching for integrable cases

4. Results for case $\ell = 1$

5. References

**Definition 1.**

The ODE system is ***locally integrable*** near a ***stationary point*** (SP) of the system if it has enough analytic integrals in a vicinity of the SP. It is evident that an integrable system is locally integrable at each of its stationary points.

**Hypothesis 1 ([Edneral, 2023]).**

*If an autonomous polynomial ODE system is locally integrable in the neighborhood of all its stationary points, then it is globally integrable.*

Therefore, to find global integrability, we must first find all stationary points of the ODE system and then find out whether the system is locally integrable in their neighborhoods.

Let $X = (p, q, r, \gamma_1, \gamma_2, \gamma_3)$, the point $X = X^0$ be a stationary point of the system (1) and

$$M = \begin{pmatrix} 0 & \dfrac{B-C}{A}r & \dfrac{B-C}{A}q & 0 & -\dfrac{z_0}{A} & \dfrac{y_0}{A} \\ \dfrac{C-A}{B}r & 0 & \dfrac{C-A}{B}p & \dfrac{z_0}{B} & 0 & -\dfrac{x_0}{B} \\ \dfrac{A-B}{C}q & \dfrac{A-B}{C}p & 0 & -\dfrac{y_0}{C} & \dfrac{x_0}{C} & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & r & -q \\ \gamma_3 & 0 & -\gamma_1 & -r & 0 & p \\ -\gamma_2 & \gamma_1 & 0 & q & -p & 0 \end{pmatrix}$$

be a matrix of the linear part of the system (1) near the SP $X^0$.

The characteristic polynomial $\chi(\lambda)$ of the matrix $M$ is

$$\chi(\lambda) = \lambda^6 + a_4\lambda^4 + a_2\lambda^2.$$

Canceling it by $\lambda^2$ we get a bi-quadratic form, and compute its ***primary discriminant*** on $\lambda^2$:

$$D_{\lambda^2}(\chi) = a_4^2 - 4a_2. \quad (3)$$

It is a rational function $D = G/H$, where $G$ and $H$ are polynomials in system parameters.

A stationary point is ***locally integrable*** [Bruno, 2007] if $a_2 < 0$ or $D_\lambda(\chi) < 0$. But this property is not satisfied for definite values of system parameters (1).

The stationary points of the EP system form one- and two-dimensional families $\mathcal{F}_j^\ell$ in $\mathbb{R}^6$. Below $j$ is the number of the family for a given value of $\ell$.

The numerator $G$ of the primary discriminant $D_\lambda(\chi)$ depends on the set $\Xi$ of parameters

$$\Xi \stackrel{\mathrm{def}}{=} \{s,\ A,\ B,\ C\} \tag{4}$$

where $s$ is the parameter along the family $\mathcal{F}_j^\ell$ and others are parameters of the system (1). Let $\xi$ be one of the parameters (4).

By $\Delta_\xi\left(\mathcal{F}_j^\ell\right)$ we denote the ***secondary discriminant*** of the numerator $G$ of the primary discriminant (3) in the parameter $\xi$.

**Hypothesis 2.**

*If near a stationary point $X^0$ of the family $\mathcal{F}_j^\ell$ with certain parameters values (4) the EP-equations are locally integrable, then at these parameter values there is at least one secondary discriminant $\Delta_\xi\left(\mathcal{F}_j^\ell\right) = 0$.*

**Remark**

The property given in Hypothesis 2 is a necessity condition for local integrability, but not a sufficient one. So, the set of parameters for which Hypothesis 2 is satisfied should be called ***potentially integrable***. Our first task is to compute all sets of potential integrability and then to select from them the cases of local or global integrability.

The case where the characteristic polynomial $\chi(\lambda)$ has zero roots can be studied using the approach from [Stolovitch, Verstringe, 2016]. But here we assume the following Hypothesis

**Hypothesis 3.**

*For EP-equations the stationary point with four zero eigenvalues is locally integrable.*

There are three ways to check the existence of the fourth integral.
1) Finding this integral in explicit form.
2) Using the ***Normal Form*** (NF) of the system near the SP: according to [Bruno, 2007, Section 5.3] the resonant terms coefficients of the NF at a resonance of order $3$, i.e., when there exists a pair of eigenvalues with ratio $2:1$, should be zero in integrable cases. They are zero in some subcases, and in other subcases they are nonzero. If resonance of order $3$ is absent, then we need to consider the resonance with the minimum possible order.
3) If in the case that the EP-equations (1) do not have any resonances, one has to consider the eigenvalues at the stationary point of a family $\mathcal{F}_j^\ell$. If all of them do not belong to the straight line that crosses the origin of the complex plane, then in the real case the normalization transformation is analytic and the system (1) is locally integrable. See [Moser, 1958] for Hamiltonian systems and [Bruno, 2007] for the EP-equations, because they can be written as a Hamiltonian system [Gashenenko (et al.), 2012] with the same set of eigenvalues.

## 3. Algorithm of searching for integrable cases (1)

Taking into account Hypothesis 1, now the search for integrable cases consists of the following steps.

**Step 1** Fix the number $\ell$ of non-zero parameters $x_0, y_0, z_0$ and find all families $\mathcal{F}_j^\ell$ of stationary points.
**Step 2** Compute primary discriminants (3) $D_\lambda(\chi)$ on the families $\mathcal{F}_j^\ell$.
**Step 3** In the family $\mathcal{F}_j^\ell$, calculate all secondary discriminants $\Delta_\xi\left(\mathcal{F}_j^\ell\right)$ of the numerators $G$ of the primary discriminants $D_\lambda(\chi)$, where $\xi \in \Xi$ defined in (4), and their factors. All irreducible factors we divide into two groups:

## 3. Algorithm of searching for integrable cases (2)

**Ist group** are factors not depending on $s$:

$$\varphi_1, \ldots, \varphi_m. \tag{5}$$

**IInd group** are factors depending on $s$:

$$\psi_1, \ldots, \psi_n.$$

For each pair $\psi_i$, $\psi_k$, $1 \leqslant i < k \leqslant n$, we compute its resultant $R_s^{ik}(A, B, C)$ over the parameter $s$ and factorize it into irreducible factors $\eta^{ik}(A, B, C)$. So, we obtain a set $\mathcal{S}_j$ of factors (5) and factors $\eta^{ik}(A, B, C)$ that are not dependent on $s$.
For their roots, all SPs of the family $\mathcal{F}_j^\ell$ are potentially locally integrable.

**Step 4** For fixed value $\ell$ and for each set

$$f_1(A,B,C), \quad f_2(A,B,C), \quad f_3(A,B,C),$$

where $f_j \subset \mathcal{S}_j$, we compute their common real roots $A_0, B_0, C_0$, satisfying the triangle inequalities (2). Such roots are potentially globally integrable cases according to Hypothesizes 1 and 2.

**Step 5** Check the values of the parameters obtained for potential integrability by computing the normal forms of the EP system near stationary points or by finding the fourth independent integral. Usually, it is sufficient to compute coefficients of the resonant monomials of the normal form in the case of the resonance of order 3, i.e., when there exists at least one pair of non-zero eigenvalues with ration $\lambda_j : \lambda_k = 2$.

1. Introduction

2. Theory

3. Algorithm of searching for integrable cases

**4. Results for case $\ell = 1$**

5. References

**Theorem 1.**

*For $\ell = 1$ EP-equations has four families of SP:*

$$\mathcal{F}_1^1 : \{p = s,\ q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0\}\,;$$

$$\mathcal{F}_2^1 : \left\{p = \frac{x_0}{k(C-A)}, q = 0, r = \frac{s}{k},\ \gamma_1 = \frac{p}{k},\ \gamma_2 = 0, \gamma_3 = \frac{r}{k}, A \neq C,\ x_0 \neq 0\right\};$$

$$\mathcal{F}_3^1 : \left\{p = \frac{x_0}{k(B-A)}, q = \frac{s}{k}, r = 0,\ \gamma_1 = \frac{p}{k},\ \gamma_2 = \frac{q}{k}, \gamma_3 = 0, A \neq B,\ x_0 \neq 0\right\};$$

$$\mathcal{F}_4^1 : \left\{p = \frac{x_0}{k(B-A)},\ \gamma_1 = \frac{p}{k},\ \gamma_2 = \frac{q}{k}, \gamma_3 = \frac{r}{k}, A \neq B = C,\ x_0 \neq 0\right\},$$

*where $s$, $q$, $r$ are parameters.*

Under the permutation

$$q \leftrightarrow r,\ \gamma_2 \leftrightarrow \gamma_3,\ B \leftrightarrow C,\ y_0 \leftrightarrow z_0,\ t \to -t \tag{6}$$

families $\mathcal{F}_3^1 \leftrightarrow \mathcal{F}_2^1$. The families $\mathcal{F}_1^1$ and $\mathcal{F}_4^1$ are invariant under automorphism.

Let us apply this approach to the case $\ell = 1$. In this case $x_0 \neq 0, y_0 = z_0 = 0$. Now we study local integrability for the families $\mathcal{F}_1^1$, $\mathcal{F}_2^1$, $\mathcal{F}_3^1$ and $\mathcal{F}_4^1$.

Starting from the "simplest" family $\mathcal{F}_1^1$ and obtaining a set of relations $\mathcal{S}(\mathcal{F}_1^1)$ between the parameters $\Xi$, we would provide the corresponding computations for other families only taking into account the relations obtained for the families processed earlier.

So, for the family $\mathcal{F}_1^1$ we have the following:

- coefficients $a_4$ and $a_2$ of the characteristic polynomial $\chi(\mathcal{F}_1^1)$ are

$$a_4 = \frac{(B+C)\,x_0 + s^2\left(A^2 - AB - AC + 2CB\right)}{BC},$$
$$a_2 = \frac{\left((A-C)s^2 + x_0\right)\left((A-B)s^2 + x_0\right)}{BC};$$

- numerator $G$ of the discriminant $D_\lambda(\chi)$ is

$$G(\mathcal{F}_1^1) = A^2\,(A-C-B)^2\,s^4 + 2\,(BA + AC - 4CB)\,(A-C-B)\,x_0\,s^2 + (B-C)^2\,x_0^2,$$

- the secondary discriminants $\Delta_\xi$ are the following

$$\begin{aligned}
\Delta_{s^2}\left(\mathcal{F}_1^1\right) &\cong (A-2C)^2(A-2B)^2(A-B-C)^6(B-C)^2A^2B^2C^2x_0^6,\\
\Delta_A\left(\mathcal{F}_1^1\right) &\cong g_{1A}(B-C)^2x_0^3B^2C^2s^{12},\\
\Delta_B\left(\mathcal{F}_1^1\right) &\cong (A-2C)^2g_{1B}x_0^2C\,s^2,\\
\Delta_C\left(\mathcal{F}_1^1\right) &\cong (A-2B)^2g_{1C}x_0^2Bs^2,
\end{aligned}$$

where

$$\begin{aligned}
g_{1A} =& 2(B+C)^3s^6+3(C-5B)(B-5C)s^4x_0+(24B+24C)s^2x_0^2+16x_0^3,\\
g_{1B} =& (A-C)s^2+x_0, \quad g_{1C}=(A-B)s^2+x_0.
\end{aligned}$$

For the family $\mathcal{F}_2^1$ we have the following.
Connection between parameter $k$ and other parameters from the set $\Xi$ is

$$(A-C)^2\,k^4 = x_0^2 + s^2\,(A-C)^2\,.$$

Numerator $G$ of the discriminant $D_\lambda(\mathcal{F}_2^1)$ is the following

$$\begin{aligned} G(\mathcal{F}_2^1) = C^4\,(A-C)^4\,(A+B-C)^2\,s^4 + 2AC\,(A-C)^2\,(A+B-C)\times \\ \times\left(2A^2B - A^2C - 6ABC + 2A\,C^2 + 5B\,C^2 - C^3\right)s^2x_0^2 + \\ + A^2\left(A^2 - 2AB - 2CA + 3CB + C^2\right)^2 x_0^4 \end{aligned}$$

and the secondary discriminants $\Delta_\xi$ are

$$\begin{aligned}
\Delta_s\left(\mathcal{F}_2^1\right) \cong & (B-C)^2 (A-2C)^4 (A-C)^{16} (A+B-C)^6 \times \\
& \times \left(A^2 - 2AB - 2AC + 3BC + C^2\right)^2 A^6 B^2 C^8 x_0^{12}, \\
\Delta_A\left(\mathcal{F}_2^1\right) \cong & (B-C)^5 h_{2A} g_{2A}^2 B^8 C^{19} x_0^{20} s^8, \\
\Delta_B\left(\mathcal{F}_2^1\right) \cong & (A-2C)^2 (A-C)^5 g_{2B} A^3 C\, x_0^4 s^2, \\
\Delta_C\left(\mathcal{F}_2^1\right) \cong & f_{13}(s, x_0, A, B) g_{2C}^2 A^{25} B^{10} x_0^{28} s^{26},
\end{aligned}$$

where

$$\begin{aligned} h_{2A} =& 16B^5C^7s^8 - 8C^5B^3\left(5B^2 - 10BC - 27C^2\right)s^6x_0^2 + \\ & + 3BC^3\left(291B^4 + 4C\,B^3 - 638B^2C^2 + 612BC^3 + 243C^4\right)s^4x_0^4 \\ & - 8BC\,(3B + C)\left(9B^3 + 249B^2C - 557BC^2 + 171C^3\right)s^2x_0^6 + 16\,(3B + C)^4\,x_0^8, \\ g_{2A} =& 2(B - C)x_0^2 + C^2(B + C)s^2, \quad g_{2B} = (4C - 3A)\,x_0^2 + C\,(A - C)^2\,s^2, \\ g_{2C} =& 8\,(2B - A)\,x_0^2 + A^2\,(A + 2B)\,s^2, \end{aligned}$$

and $f_{13}(s, x_0, A, B)$ is a very cumbersome expression. Similar results for the family $\mathcal{F}_3^1$ can be obtained by permutation (6).

## Local integrability for the family $\mathcal{F}_4^1$

For family $\mathcal{F}_4^1$ of SP, the coefficient $a_2$ of the characteristic polynomial $\chi(\mathcal{F}_4^1)$ is equal to zero. So, we have four zero eigenvalues here. Jordan form of matrix $M(\mathcal{F}_4^1)$ has only one Jordan block $2 \times 2$, and Hypothesis 3 is applicable, and gives local integrability for the family $\mathcal{F}_4^1$.

Collecting together the previous results of local integrability for the families $\mathcal{F}_j^1$, $j = 1{,}2{,}3$, we obtain

$$\begin{aligned}
\mathcal{F}_1^1 :& L_{11} = \{B = C\}, L_{12} = \{A = 2C\}, L_{13} = \{A = 2B\}, L_{14} = \{A = B + C\}; \\
\mathcal{F}_2^1 :& L_{21} = \{B = C\}; L_{22} = \{A = 2C\}, L_{23} = \{A = C\}, L_{24} = \{C = A + B\}; \\
& L_{25} = \left\{A^2 - 2AB - 2AC + 3BC + C^2 = 0\right\}. \\
\mathcal{F}_3^1 :& L_{31} = \{B = C\}, L_{32} = \{A = 2B\}, L_{33} = \{A = B\}, L_{34} = \{B = A + C\}, \\
& L_{35} = \left\{A^2 - 2AB - 2AC + 3BC + B^2 = 0\right\},
\end{aligned} \tag{7}$$

- For all families $\mathcal{F}_j^1$, $j = 1,2,3$, there is a case $L_{11} \equiv L_{21} \equiv L_{31}$, i.e. $B = C$. So, it is an integrable case.
- For families $\mathcal{F}_1^1$ and $\mathcal{F}_2^1$, there are cases $L_{12} \equiv L_{22}$, i.e. $A = 2C$, and for family $\mathcal{F}_3^1$ there is a case $L_{33}$, i.e. $A = B$. So, case $A = B = 2C$ is integrable; similarly, case $A = C = 2B$ is also integrable.

There is one more case $L_{13}$, $L_{24}$, $L_{32}$, that is, $A = 2B$, $C = 3B$ or

$$A = 2B, \quad B = C/3. \tag{8}$$

Here an additional integral is absent because there are non-zero resonant terms in the NF

Any other combinations of local integrability cases contradict the triangle inequalities or give the case $B = C$. These cases of integrability are known.

In the above, we considered only factors that do not depend on $s$. Let us take into account factors, which do depend on mentioned above parameters:

$$\begin{aligned}
&\mathcal{F}_1^1 \colon g_{1A},\ g_{1B},\ g_{1C};\\
&\mathcal{F}_2^1 \colon g_{2A},\ g_{2B},\ g_{2C},\ h_{2A},\ f_{13}(s, x_0, A, B);\\
&\mathcal{F}_3^1 \colon g_{3A},\ g_{3B},\ g_{3C},\ h_{3A},\ f_{13}(s, x_0, A, C),
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
g_{3A}(B,C) = g_{2A}(C,B),\ g_{3B}(B,C) = g_{2C}(C,B),\\
g_{3C}(B,C) = g_{2B}(C,B),\ h_{3A}(B,C) = h_{2A}(C,B).
\end{aligned}$$

### Remark

The parameter $s$ along one family is independent of the parameters along the other ones. So, it can be eliminated from the pairs of factors related to the same family of SP only.

### Brief description of the computational procedure

**Step 1** Using the elimination technique, we prepare sets of polynomial factors for each family $\mathcal{F}_j^1$, $j = 1{,}2{,}3$, eliminating parameters $s$ from the polynomials in Formula (9).

**Step 2** The sets $\mathcal{S}_j$ of possible factors are prepared by union polynomials from Formula (7) with factors obtained in the previous step.

**Step 3** From each factor of the corresponding set $\mathcal{S}_j$, $j = 1{,}2{,}3$, we construct a system of polynomial equations, compute the Gröbner basis for it and find all real non-trivial solutions, that is, all parameters $A, B, C$ are non-zero.

**Step 4** The solutions obtained are checked for whether they satisfy the triangle inequalities (2).

## 4. Results for case $\ell = 1$. Some new potentially integrable cases (3)

In Step 2 we prepare 3 sets $\mathcal{S}_j$, $j = 1,2,3$, which contain all possible factors for each family, namely 6 factors in $\mathcal{S}_1$ and 14 factors in each $\mathcal{S}_2$ and $\mathcal{S}_3$ sets. In total, we get $6 \cdot 14 \cdot 14 = 1176$ possible combinations of factors.

Implementation of Steps 3 and 4 gave us 17 non-trivial solutions, i.e., solutions with non-zero values of $A, B, C$ satisfying the triangle inequalities (2). 9 solutions of them correspond to Case 2 of Lagrange–Poisson: common case and 8 particular cases. 2 solutions correspond to Case 3 of Kovalevskaya up to the permutation (6) $B \leftrightarrow C$, $q \leftrightarrow r$. The 2 solutions correspond to the case $A = 2B, B = C/3$ considered above. So, we could find 2 new pairs of solutions:

1) $A = 2C$, $B = 3C/2$ and symmetrical

$$A = 2B, \quad B = 2C/3. \tag{10}$$

# 4. Results for case $\ell = 1$. Some new potentially integrable cases (4)

2) $A = 2C$, $3B^3 - 6B^2C + 5B\,C^2 - 4C^3 = 0$ and symmetrical under permutation $B \leftrightarrow C$ is $A = 2B$, $4B^3 - 5B^2C + 6B\,C^2 - 3C^3 = 0$. The last homogeneous cubic equation has only one real root for

$$B/C = (\alpha - 47/\alpha + 5)\,/12 \approx 0.67456267248393436447, \tag{11}$$

where $\alpha = \sqrt[3]{233 + 36\sqrt{122}}$.

Among the set of roots obtained, we need to select only those that satisfy the triangle inequalities, which are successful on the following values $B/C$ only:

$$B/C \in \{0.6656638025, \quad 0.6668031182, \quad 0.9967129831\}. \tag{12}$$

## 4. Results for case $\ell = 1$. Some new potentially integrable cases (5)

Summarizing the results of potentially integrable cases (8), (10), (11), (12) we can state that all the cases are implemented under the conditions $A = 2B$ and $B/C = \beta$, where the parameter $\beta$

$$\beta \in \{1/3,\ 0.6656638025,\ 2/3,\ 0.6668031182,\ 0.6745626725,\ 0.9967129831\}.$$

For each of these values, we need to provide the check of local integrability separately for the families $\mathcal{F}_1^1$, $\mathcal{F}_2^1$, and $\mathcal{F}_3^1$. Here we encounter three different situations concerning ***Resonant condition*** for the definite values of $\beta$.

1) Resonance of order 3 is implemented for some real values of the parameter $x_0$.

2) Resonance of order 3 does not exist for any real values of the parameter $x_0$, but is implemented for some resonances of higher order.

3) There are no resonances of any orders for any real values of the parameter $x_0$.

## 4. Results for case $\ell = 1$. Some new potentially integrable cases (6)

**Theorem 2.**

*For the case $A = 2B$, $B/C = \beta$, $y_0 = z_0 = 0$, and $1/3 \leqslant \beta \leqslant 1$, the EP-equations are locally integrable in the vicinity of the family $\mathcal{F}_1^1$ of the stationary points.*

We collect the ***minimal order of resonance*** existing for the real values of the parameters $s, x_0$

**Table 1:** Minimal order of a resonance existence for real values of parameters $s, x_0$

| No | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| $\beta$ | 1/3 | 0.6656638025 | 2/3 | 0.6668031182 | 0.6745626725 | 0.9967129831 |
| $\mathcal{F}_2^1$ | 3 | – | – | 3457 | 62 | 3 |
| $\mathcal{F}_3^1$ | 3 | 4 | 4 | 4 | 4 | 26 |

For Situation|1), we select the values No 1 and No 6, when the resonance of the order $3$ exists. The computational procedure here is similar to the procedure described for the family $\mathcal{F}_1^1$ above.

**Theorem 3.**

*For $\beta$ values No 1 and No 6 in Table 1 the NFs in the vicinity of the family $\mathcal{F}_2^1$ are not integrable.*

Now we check the values No 2 and 3 for the family $\mathcal{F}_2^1$, where Situation 3) takes place.

**Theorem 4.**

*For $\beta$ values No 2 and No 3 in Table 1 the corresponding NFs in the vicinity of the family $\mathcal{F}_2^1$ are integrable.*

Finally, we give the description of results of the NF computation for the family $\mathcal{F}_3^1$ for values No 2, 3, 4, 5, i.e. when Situation|2) takes place with resonance of order 4.

For the value $\beta = 2/3$ (No 3 in Table 1) the coefficients of the resonant terms can be computed symbolically, but for other values (No 2,4,5) only numerically. The coefficients obtained, divided by $\sqrt{s}$, are presented in Table 2.

**Table 2:** Table of the resonant terms coefficients. Here $\delta = i42^{3/4}\sqrt{6}/72$.

| No | $c_{123}$ | $c_{214}$ | $c_{310}$ | $c_{420}$ |
|---|---|---|---|---|
| 2 | $-0.192267822892i$ | $0.192267822892i$ | $0.0413172384893i$ | $-0.0413172384893i$ |
| 3 | $-\delta$ | $\delta$ | $\delta/3$ | $-\delta/3$ |
| 4 | $-0.192107258655$ | $-0.192107258655$ | $-0.0413243675971$ | $-0.0413243675971$ |
| 5 | $-0.190226341400$ | $-0.190226341400$ | $-0.0411825670879$ | $-0.0411825670879$ |

**Theorem 5.**

*For each value $\beta$ No 2, 3, 4, 5 in Table 1 of the family $\mathcal{F}_3^1$ the NF has non-zero resonant monomials and therefore is not locally integrable.*

So, now we can summarize all the computations for the case $\ell = 1$.

**Theorem 6.**

*All known cases of global integrability were obtained. For all new computed potentially integrable cases of the form $A = 2B$, $B = \beta C$, with $\beta \in \mathcal{B}$, there are no global integrability.*

We have some new results for cases $\ell = 2$ and $\ell = 3$, but we do not have time to present them.

1. Introduction

2. Theory

3. Algorithm of searching for integrable cases

4. Results for case $\ell = 1$

5. References

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*THANKS FOR ATTENTION*