Deciding Summability via Residues in Theory and in Practice

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Motivation: Rational Integrability via Residues (1 of 2)

Let $\mathbb{K}$ be a field of characteristic 0, and consider a rational function $f(x) \in \mathbb{K}(x)$. There is a complete partial fraction decomposition

$$f(x) = p(x) + \sum_{k \geqslant 1} \sum_{\alpha \in \overline{\mathbb{K}}} \frac{c_k(\alpha)}{(x-\alpha)^k} \qquad (*)$$

where $p(x) \in \mathbb{K}[x]$ is a polynomial and $c_k(\alpha) \in \overline{\mathbb{K}}$ (almost all 0), and where $\overline{\mathbb{K}}$ denotes an algebraic closure of $\mathbb{K}$.

Then we know that $f(x)$ is *rationally integrable*, i.e., there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g'(x)$, if and only if the *residues*

$$\operatorname{res}(f, \alpha, 1) := c_1(\alpha) = 0 \qquad \text{for every } \alpha \in \overline{\mathbb{K}}.$$

Problem: the actual computation of the decomposition $(*)$ is too expensive (or maybe impossible). How to compute the residues $\operatorname{res}(f, \alpha, 1) = c_1(\alpha)$ without computing $(*)$?

Motivation: Rational Integrability via Residues (2 of 2)

Better question: how to efficiently compute a <u>$\mathbb{K}$-rational representation</u> of the (first-order) residues $c_1(\alpha)$ of $f(x) \in \mathbb{K}(x)$?

The so-called Hermite Reduction (first discovered by Ostrogradskii)

$$f(x) = g'(x) + h(x),$$

where $g(x), h(x) \in \mathbb{K}(x)$ such that $h(x)$ is proper and has squarefree monic denominator (only simple poles).

For such $h(x) = a(x)/b(x) \neq 0$ the unique polynomial $r(x) \in \mathbb{K}[x]$ with $\deg\big(r(x)\big) < \deg\big(b(x)\big)$ such that

$$r(x)b'(x) \equiv a(x) \pmod{b(x)}$$

satisfies $r(\alpha) = c_1(\alpha)$ for each root $\alpha \in \overline{\mathbb{K}}$ of $b(x) \in \mathbb{K}[x]$.

Thus we obtain the pair $\big(b(x), r(x)\big) \equiv \Big\{\big(\alpha, c_1(\alpha)\big) \;\Big|\; \alpha \in \overline{\mathbb{K}}\Big\}$, using only linear algebra, differentiation, and gcd computations.

The Summability Problem in the Shift Case

Consider now the *forward difference operator*

$$\Delta_1 : \mathbb{K}(x) \to \mathbb{K}(x) : g(x) \mapsto g(x+1) - g(x).$$

This can be considered as a discretization of the differentiation $g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$: set $h = 1$ instead of taking the limit.

The discrete analogue of the rational integrability problem is the *rational summability problem*: for a given $f(x) \in \mathbb{K}(x)$, decide whether (just yes/no) there exists $g(x) \in \mathbb{K}(x)$ such that

$$f(x) = g(x+1) - g(x).$$

If $f(x) \in \mathrm{im}(\Delta_1)$ as above, we say $f(x)$ is *rationally summable*.

Discrete Residues in the Shift Case

Let us rearrange the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = p(x) + \sum_{k \geqslant 1} \sum_{\omega \in \overline{\mathbb{K}}/\mathbb{Z}} \sum_{\alpha \in \omega} \frac{c_k(\alpha)}{(x-\alpha)^k},$$

where $\overline{\mathbb{K}}/\mathbb{Z}$ is the set of *orbits* $\omega(\alpha)$ for α ranging over $\overline{\mathbb{K}}$, where

$$\omega(\alpha) := \alpha + \mathbb{Z} = \{\alpha + n \mid n \in \mathbb{Z}\}.$$

The *discrete residue* of $f(x) \in \mathbb{K}(x)$ at the orbit $\omega \in \overline{\mathbb{K}}/\mathbb{Z}$ of order k is defined by the finite sum

$$\mathrm{dres}(f, \omega, k) := \sum_{\alpha \in \omega} c_k(\alpha).$$

Proposition (Chen-Singer in Adv. Appl. Math.'12)

The rational function $f(x) \in \mathbb{K}(x)$ is rationally summable if and only if $\mathrm{dres}(f, \omega, k) = 0$ for each orbit $\omega \in \overline{\mathbb{K}}/\mathbb{Z}$ and order $k \in \mathbb{N}$.

$\mathbb{K}$-Rational Representations of Shift Discrete Residues

A set of pairs of polynomials

$$\big(B_1(x), D_1(x)\big), \ldots, \big(B_m(x), D_m(x)\big) \in \mathbb{K}[x] \times \mathbb{K}[x]$$

is called a *$\mathbb{K}$-rational system of discrete residues* for $f(x) \in \mathbb{K}(x)$ if:

1. $f(x)$ has no poles of order greater than m;
2. for each $\alpha \in \overline{\mathbb{K}}$ such that $B_k(\alpha) = 0$, the evaluation $D_k(\alpha) = \mathrm{dres}\big(f, \omega(\alpha), k\big)$; and
3. for each $\omega \in \overline{\mathbb{K}}/\mathbb{Z}$ and $k \in \mathbb{N}$ such that $\mathrm{dres}(f, \omega, k) \neq 0$, there exists precisely one $\alpha \in \omega$ such that $B_k(\alpha) = 0$.

In short, each polynomial $B_1(x), \ldots, B_m(x)$ encodes *where* $f(x)$ has possibly non-zero residues, and the corresponding polynomials $D_1(x), \ldots, D_m(x)$ encode *what* these residues are.

One can prove theoretically, using Galois theory, that such $\mathbb{K}$-rational systems of discrete residues always exist.

But even better, we can compute them! (A.-Sitaula in ISSAC'24)

Reducing to Case of Simple Poles

Suppose $f(x) \in \mathbb{K}(x)$ is proper, with partial fraction decomposition

$$f(x) = \sum_{k \geqslant 1} \sum_{\alpha \in \overline{\mathbb{K}}} \frac{c_k(\alpha)}{(x-\alpha)^k}.$$

We wish to compute $f_k(x) \in \mathbb{K}(x)$ such that

$$f_k(x) = \sum_{\alpha \in \overline{\mathbb{K}}} \frac{c_k(\alpha)}{x-\alpha},$$

because then we immediately have

$$\mathrm{dres}(f, \omega, k) = \mathrm{dres}(f_k, \omega, 1)$$

for every $\omega \in \mathbb{K}$, which allows us to reduce to the case where f has only simple poles.

Iterated Hermite Reduction

Suppose $f(x) \in \mathbb{K}(x)$ is proper. Then recall

$$\texttt{HermiteReduction}(f) = (g, h)$$

for the unique $g, h \in \mathbb{K}(x)$ such that h has only simple poles and

$$f = g' + h.$$

Our first step is to iterate this process to obtain $\texttt{HermiteList}(f)$:

Input: A proper rational function $0 \neq f(x) \in \mathbb{K}(x)$.
Output: The list $\big(f_1(x), \ldots, f_m(x)\big)$ with $f_k = \sum_\alpha c_k(\alpha)/(x - \alpha)$.

Initialize loop: $m \leftarrow 0$; $g \leftarrow f$;
while $g \neq 0$ **do**
  $(g, \hat{f}_{m+1}) \leftarrow \texttt{HermiteReduction}(g)$;
  $m \leftarrow m + 1$;
end while;
$f_k \leftarrow (-1)^{k-1}(k-1)!\hat{f}_k$;
return $(f_1, \ldots, f_m)$.

Simple Reduction (1 of 2)

Using `HermiteList`, we can assume the proper, reduced $f(x) = a(x)/b(x)$ has squarefree $b(x) \in \mathbb{K}[x]$.

Our next task is to compute a *reduced form* $\bar{f}(x) \in \mathbb{K}(x)$ having precisely one pole $\alpha \in \overline{\mathbb{K}}$ belonging to each orbit $\omega \in \overline{\mathbb{K}}/\mathbb{Z}$ such that $\mathrm{dres}(f, \omega, 1) \neq 0$ (and poles nowhere else), and such that

$$\bar{f}(x) = \frac{\bar{a}(x)}{\bar{b}(x)} = \sum_{\alpha} \frac{\mathrm{dres}(f, \omega(\alpha), 1)}{x - \alpha}.$$

Then we can take $B(x) = \bar{b}(x)$, and obtain $D(x) = r(x)$ for the unique polynomial $r(x) \in \mathbb{K}[x]$ with $\deg\big(r(x)\big) < \big(\bar{b}(x)\big)$ such that $r(x)\bar{b}'(x) \equiv \bar{a}(x) \pmod{\bar{b}(x)}$ like before.

This $\big(B(x), D(x)\big)$ is a $\mathbb{K}$-rational representation of the discrete residues of $f(x)$.

Interlude: Partial Partial Fraction Decompositions

Suppose $f(x) \in \mathbb{K}(x)$ is proper with squarefree denominator $b(x)$. Computing the complete partial fraction decomposition

$$f(x) = \sum_{\alpha \in \overline{\mathbb{K}}} \frac{c_1(\alpha)}{x - \alpha}$$

is impossible in general, but only because we don't know how to compute the complete factorization $b(x) = \prod_{\ell=1}^{n}(x - \alpha_\ell)$.

However, if we are already given a factorization $b(x) = \prod_{\ell=1} b_\ell(x)$ with $b_\ell(x) \in \mathbb{K}[x]$, it is easy and efficient to compute the unique $a_\ell(x) \in \mathbb{K}[x]$ with every $\deg\big(a_\ell(x)\big) < \deg\big(b_\ell(x)\big)$ such that

$$f(x) = \sum_{\ell=1}^{n} \frac{a_\ell(x)}{b_\ell(x)}.$$

We write

$$\texttt{ParFrac}\big(f; (b_1, \ldots, b_n)\big) := (a_1, \ldots, a_n).$$

Interlude: Shift Set

For a non-constant polynomial $b(x) \in \mathbb{K}[x]$, it was observed by Abramov in 1971 that the

$$\mathrm{ShiftSet}(b) := \left\{ n \in \mathbb{N} \;\middle|\; \gcd(b(x), b(x+n)) \neq 1 \right\}$$

is the set of positive integer roots of

$$\mathrm{Res}_x\big(b(x+z), b(x)\big) = R(z) \in \mathbb{K}[z].$$

We observe that this is also the set of positive integer roots of

$$\tilde{R}(z) = \frac{R(z)}{z \cdot \gcd(R(z), R'(z))}.$$

Depending on $\mathbb{K}$, there are now more efficient ways to compute this set (see Gerhard-Giesbrecht-Storjohann-Zima in ISSAC'03 and Man-Wright in ISSAC'94).

Simple Reduction (2 of 2)

Input: Proper $f(x) = a(x)/b(x) \in \mathbb{K}(x)$ with squarefree $b(x)$.
Output: Reduced form $\bar{f}(x)$ of $f(x)$.

$S \leftarrow \texttt{ShiftSet}(b)$;
if $S = \varnothing$ **then**
  $\bar{f}(x) \leftarrow f(x)$;
else
  for $\ell \in S$ **do**
    $g_\ell(x) \leftarrow \gcd\big(b(x), b(x-\ell)\big)$;
  end for;
  $b_0(x) \leftarrow b(x)/\mathrm{lcm}(g_\ell(x) \mid \ell \in S)$; ▻ Exact division.
  for $\ell \in S$ **do**
    $b_\ell(x) \leftarrow gcd\big(b_0(x-\ell), b(x)\big)$;
  end for;
  $(a_\ell \mid \ell \in S) \leftarrow \texttt{ParFrac}\big(f; (b_\ell \mid \ell \in S)\big)$;
  $\bar{f}(x) \leftarrow \sum_{\ell \in N} \frac{a_\ell(x+\ell)}{b_\ell(x+\ell)}$;
end if;
return $\bar{f}$.

Complete Algorithm to Compute Shift Discrete Residues

Input: A proper rational function $0 \neq f \in \mathbb{K}(x)$.
Output: A $\mathbb{K}$-rational system of discrete residues
$\big(B_1(x), D_1(x)\big), \ldots, \big(B_m(x), D_m(x)\big) \in \mathbb{K}[x] \times \mathbb{K}[x]$ for $f(x)$.

$(f_1, \ldots, f_m) \leftarrow \texttt{HermiteList}(f)$;
for $k = 1..m$ **do**
$\quad \bar{f}_k \leftarrow \texttt{SimpleReduction}(f_k)$;
$\quad (B_k, D_k) \leftarrow \texttt{FirstResidues}(\bar{f}_k)$;
end for;
return $\big((B_1, D_1), \ldots, (B_m, D_m)\big)$.

Example (1 of 3)

$$f := \frac{1}{x^3(x+2)^3(x+3)(x^2+1)(x^2+4x+5)^2} = \sum_{k=1}^{3} \sum_{\alpha \in \overline{\mathbb{K}}} \frac{c_k(\alpha)}{(x-\alpha)^k}.$$

$\texttt{HermiteList}(f) = (f_1, f_2, f_3)$; certified: $f_k = \sum_\alpha c_k(\alpha)(x-\alpha)^{-1}$.

Explicitly:

$$f_1 = \frac{787x^5 + 4803x^4 + 9659x^3 + 9721x^2 + 9502x + 5008}{18000(x^2+1)(x+3)(x^2+4x+5)(x+2)x};$$

$$f_2 = -\frac{787x^3 + 3372x^2 + 4696x + 1030}{18000(x^2+4x+5)x(x+2)};$$

$$f_3 = -\frac{7x-1}{300(x+2)x}.$$

We do the rest of the algorithm on f_1 only; the rest are easier.

Example (2 of 3)

$$f_1 = \frac{787x^5 + 4803x^4 + 9659x^3 + 9721x^2 + 9502x + 5008}{18000(x^2+1)(x+3)(x^2+4x+5)(x+2)x}.$$

Denominator: $b(x) = (x^2+1)(x+3)(x^2+4x+5)(x+2)x$ has

$$\texttt{ShiftSet}(b) = \{1, 2, 3\}.$$

$b_0(x) = (x+3)(x^2+4x+5);$	leftmost zeros
$b_1(x) = x+2;$	zeros one step right
$b_2(x) = x^2+1;$	zeros two steps right
$b_3(x) = x.$	rightmost zeros

Then $f_1(x) = \frac{a_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3}$, and the reduced form

$$\bar{f}_1(x) = \frac{a_0(x)}{b_0(x)} + \frac{a_1(x+1)}{b_1(x+1)} + \frac{a_2(x+2)}{b_2(x+2)} + \frac{a_3(x+3)}{b_3(x+3)}$$

$$= \frac{273x + 1387}{20000(x+3)(x^2+4x+5)}$$

Example (3 of 3)

Our algorithm produces (B_1, D_1) in lieu of the first-order discrete residues of f, with

$$B_1(x) = (x+3)(x^2+4x+5); \qquad \text{and}$$
$$D_1(x) = \tfrac{59}{16000}x^2 + \tfrac{33}{40000}x - \tfrac{1321}{80000}.$$

One can check that

$$c_1(f,0) = \tfrac{313}{33750}; \quad c_1(f,-2) = \tfrac{1}{250}; \quad c_1(f,-3) = \tfrac{1}{1080};$$

$$c_1(f,\pm\sqrt{-1}) = \tfrac{\pm\sqrt{-1}-7}{16000}; \quad c_1\big(f,\pm\sqrt{-1}-2\big) = \tfrac{\mp 1119\sqrt{-1}-533}{80000}.$$

And indeed:

$$\mathrm{dres}(f,\omega(0),1) = 71/5000 = D_1(-3);$$
$$\mathrm{dres}(f,\omega(\pm\sqrt{-1}),1) = \tfrac{\mp 557\sqrt{-1}-284}{40000} = D_1(\pm\sqrt{-1}-2).$$

General Summability Problem and Obstructions

A *difference field* is a pair $(\mathcal{M}, \sigma)$ consisting of a field $\mathcal{M}$ equipped with an endomorphism σ such that the subfield of *σ-invariants* $\mathbb{K} := \mathcal{M}^{\sigma} = \{c \in \mathcal{M} \mid \sigma(c) = c\}$ is relatively algebraically closed inside of $\mathcal{M}$.

The corresponding *forward difference operator* is $\Delta := \sigma - \mathrm{id}_{\mathcal{M}}$, so that $\Delta(g) = \sigma(g) - g$ for $g \in \mathcal{M}$.

The *summability problem* for $(\mathcal{M}, \sigma)$ is to be able to decide, for a given $f \in \mathcal{M}$, whether (just yes/no) there exists $g \in \mathcal{M}$ such that $f = \sigma(g) - g = \Delta(g)$. If so, we say f is *summable* (in $\mathcal{M}$).

Since Δ is a $\mathbb{K}$-linear endomorphism of the $\mathbb{K}$-vector space $\mathcal{M}$, there exist many (abstract) $\mathbb{K}$-linear maps ρ from $\mathcal{M}$ to other $\mathbb{K}$-vector spaces such that $\ker(\rho) = \mathrm{im}(\Delta)$. We call any such map a (complete) *$\mathbb{K}$-linear obstruction to summability*.

Examples of Difference Fields Arising in Practice

shift: $\mathcal{M} = \mathbb{K}(x)$; and $\sigma : g(x) \mapsto g(x+1)$.

q-dilation: $\mathcal{M} = \mathbb{K}(x)$; and $\sigma : g(x) \mapsto g(qx)$, where $q \in \mathbb{K}$ is not 0 and not a root of unity.

Mahler: $\mathcal{M} = \mathbb{K}(x)$; and $\sigma : g(x) \mapsto g(x^m)$, where $m \in \mathbb{Z}_{\geqslant 2}$.

elliptic shift: $\mathcal{M}$ is the field of rational functions on an elliptic curve $\mathcal{E}$ over $\mathbb{K}$; and $\sigma : g(x) \mapsto g(x \oplus t)$, where $t \in \mathcal{E}(\mathbb{K})$ is a non-torsion point, and $\oplus$ denotes the elliptic group law.

elliptic Mahler: $\mathcal{M}$ is the field of rational functions on an elliptic curve $\mathcal{E}$ over $\mathbb{K}$; and $\sigma : g(x) \mapsto g(m.x)$, where $m \in \mathbb{Z}_{\geqslant 2}$, and $m.x$ is the multiplication-by-m map under the elliptic group law.

Discrete Residues in the q-Dilation Case

Let us rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = \ell(x) + \sum_{k \geqslant 1} \sum_{\omega \in \overline{\mathbb{K}}^{\times}/q^{\mathbb{Z}}} \sum_{\alpha \in \omega} \frac{c_k(\alpha)}{\left(1 - \alpha^{-1}x\right)^k},$$

where $\ell(x) = \sum_{j \in \mathbb{Z}} \ell_j x^j \in \mathbb{K}[x, x^{-1}]$ is a Laurent polynomial, and $\overline{\mathbb{K}}^{\times}/q^{\mathbb{Z}}$ is the set of *orbits* $\omega(\alpha)$ for α ranging over $\overline{\mathbb{K}}^{\times}$, where

$$\omega(\alpha) := \alpha \cdot q^{\mathbb{Z}} = \{\alpha q^n \mid n \in \mathbb{Z}\}.$$

The *q-discrete residue* of $f(x) \in \mathbb{K}(x)$ at the orbit $\omega \in \overline{\mathbb{K}}^{\times}/q^{\mathbb{Z}}$ of order k is defined by the finite sum

$$q\text{-}\mathrm{dres}(f, \omega, k) := \sum_{\alpha \in \omega} c_k(\alpha); \qquad \text{and} \qquad q\text{-}\mathrm{dres}(f, \infty) := \ell_0.$$

Proposition (Chen-Singer in Adv. Appl. Math.'12)

The rational function $f(x) \in \mathbb{K}(x)$ is rationally q-summable if and only if $\mathrm{dres}(f, \omega, k) = 0$ for every $\omega \in \overline{\mathbb{K}}^{\times}/q^{\mathbb{Z}}$ and every $k \in \mathbb{N}$ and $q\text{-}\mathrm{dres}(f, \infty) = 0$.

Obstructions to Summability in the Other Cases

In recent years, analogous obstructions to summability have been developed in other cases (but requiring a few more technical details to state precisely).

- In the Mahler case, *Mahler discrete residues* were developed (A.-Zhang in ISSAC'22), giving a complete obstruction to $f(x) = g(x^m) - g(x)$ for some $g(x) \in \mathbb{K}(x)$.
- Still in the Mahler case, for given $\lambda \in \mathbb{Z}$, more general *λ-twisted Mahler discrete residues* were developed (A.-Zhang in IMRN'24), giving a complete obstruction to $f(x) = m^\lambda g(x^m) - g(x)$ for some $g \in \mathbb{K}(x)$.
- In the elliptic shift case, *orbital residues* were developed (Dreyfus-Hardouin-Roques-Singer in Inventiones'18), giving a partial obstruction to $f(x) = g(x \oplus t) - g(x)$.
- Still in the elliptic case, a pair of *panorbital residues* were introduced (Babbitt, UT Dallas Ph.D. thesis '25), completing the orbital residue obstruction introduced by DHRS in 2018.